PHYS-4601 Homework 17 Due 3 Mar 2016

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **White Dwarfs** *based on Griffiths 5.35*

   White dwarfs are old, dead stars that don't support nuclear fusion any more. Instead, electron degeneracy pressure keeps them from collapsing into black holes. In this problem, you'll work out the white dwarf radius by finding the minimum energy. *Note:* You will need equation [5.45] from Griffiths.

   (a) Imagine that the white dwarf is a sphere of uniform density. Caculate the gravitational potential energy V in terms of its radius R, total mass M, and Newton's constant G. *Hint*: To do this, work out the potential energy of a shell of thickness dr at radius r, which comes from the sphere of matter inside that shell. Then integrate from $r = 0 \to R$. You should find $V = -3GM^2/5R$.

   (b) Now treat the electrons as a free electron gas. Write their total energy (which is kinetic) in terms of M, R, the electron mass m, the nucleon mass m_N, the number of electrons per nucleon q, and $\hbar$. Then add this to the result of part (a) to get the total energy and find the radius R where the energy is minimized.

   (c) Assuming $q \approx 1/2$ (the average nucleus is a helium nucleus), find the radius (in km) of a white dwarf with the mass of the sun. You will need $G = 7\times10^{-11}$ m^3/kg/s^2, $\hbar = 1\times10^{-34}$ Js, $m = 9 \times 10^{-31}$ kg, $m_N = 2 \times 10^{-27}$ kg, and $M = 2 \times 10^{30}$ kg.

   (d) A neutron star is an extremely dense star where all the matter is neutrons, which is supported against collaps by the degeneracy pressure of the neutrons. In this case, $q = 1$. Find the ratio of the radius of a neutron star to the radius of a white dwarf of the same mass (assume $q \approx 1$ for the white dwarf), both in terms of physical constants and as a pure number to 1 significant digit.

2. **Bloch's Theorem**

   In class, we stated Bloch's theorem as saying that any stationary state wavefunction of a periodic potential $V(x + a) = V(x)$ can be written to satisfy the condition

   $$\psi(x + a) = e^{iKa}\psi(x) \ (K \text{ real}) \ . \tag{1}$$

   Alternately, Bloch's theorem can state that any stationary state wavefunction of the same potential can be written as

   $$\psi(x) = e^{iKx}u(x) \text{ where } u(x + a) = u(x) \ . \tag{2}$$

   (a) Show that these two formulations are equivalent (that is, show that if ψ satisfies (1) then it satisfies (2) and vice-versa).

   (b) If you write $u(x) = \sum_q c_q e^{iqx}$ as a Fourier series, what are the allowed values of q? Therefore, what are the allowed values of the momentum for a wavefunction with a given K?

3. **Specific Heat of the Free-Electron (Fermi) Gas**

In this problem, you'll explore the thermal properties of the electrons in a simple metal. Recall that the total number of electrons and total energy of the gas (on average) are

$$N = \sum_{states} f\left(\frac{\epsilon-\mu}{T}\right) \ , \quad U = \sum_{states} \epsilon f\left(\frac{\epsilon-\mu}{T}\right) \ , \quad f(x) = \frac{1}{e^x+1} \ . \tag{3}$$

Here, T is the temperature, and μ is the chemical potential. For free electrons in volume V, the energy of each state is $\epsilon = \hbar^2k^2/2m$ in terms of the wavevector magnitude k, and there are $(Vk^2/\pi^2)dk$ states at that energy. It is a good approximation to replace the sum over states with integrals to define the number and energy densities

$$n = \frac{1}{\pi^2}\int_0^\infty dk\, k^2 f(x) \ , \quad \rho = \frac{\hbar^2}{2m\pi^2}\int_0^\infty dk\, k^4 f(x) \ , \quad x = \frac{1}{T}\left(\frac{\hbar^2k^2}{2m}-\mu\right) \ . \tag{4}$$

In this problem, you will find the specific heat of the free electron gas at low temperatures, which is defined as

$$c_V = \frac{d\rho}{dT} \text{ with } \frac{dn}{dT} = 0 \ . \tag{5}$$

Work only to lowest order in the temperature.

(a) Argue that you can approximate

$$\int_0^\infty dk\, k^2 g(x) = \frac{T}{2\mu}\left(\frac{2m\mu}{\hbar^2}\right)^{3/2}\int_{-\infty}^\infty dx\, g(x) \tag{6}$$

for $x = (\epsilon-\mu)/T$ and any function $g(x)$ as $T \to 0$. Be careful with limits of integration.

(b) Use conservation of particle density $dn/dT = 0$ to show that $d\mu/dT \to 0$ as $T \to 0$. Hint: Use your previous results and also the fact that df/dx is an even function of x.

(c) Show that

$$c_V = \frac{mT}{\hbar^3\pi^2}\sqrt{2m\mu}\int_{-\infty}^\infty dx x^2 \frac{df}{dx} \ , \tag{7}$$

where the x integral is a pure number. To lowest order, you may set $d\mu/dT \to 0$ as follows from part (b).

(d) Finally, at $T = 0$, $\mu = E_F$, the Fermi energy, as discussed in class. Therefore argue that $c_V \propto Tn^{1/3}$.

4. **Bose-Einstein Condensation** *from Griffiths 5.29*

Consider the Bose-Einstein distribution for spin-0 bosons. As for fermions (see equation (4)), we can usually approximate the sum over states as an integral over wavevectors:

$$n = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 f(x) \ , \quad \rho = \frac{\hbar^2}{4m\pi^2}\int_0^\infty dk\, k^4 f(x) \ , \quad x = \frac{1}{T}\left(\frac{\hbar^2k^2}{2m}-\mu\right) \ , \tag{8}$$

where the factor of two difference is due to the fact that there is only one spin state and where now f is the Bose-Einstein distribution $f(x) = 1/(e^x-1)$.

Consider the case that $\mu = 0$. If there is a fixed number density n of bosons, this occurs at a fixed critical temperature T_c; $\mu < 0$ for higher temperatures, and the integral approximation above fails for lower temperatures because all the bosons go into the ground state (*Bose-Einstein condensation*). Find T_c as a function of n. *Hint:* the integral

$$\int_0^\infty dx \, \frac{x^{s-1}}{e^x - 1} = \Gamma(s)\zeta(s) \ , \tag{9}$$

in terms of the gamma function and Riemann zeta function. You may write your answer in terms of these functions (that is, you don't need to give the numerical value for those functions).