PHYS-4601 Homework 16 Due 25 Feb 2016

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **The Density Matrix/Operator**

   A system in a well-defined quantum state $|\psi\rangle$ (ie, a linear superposition of some basis kets) is said to be in a *pure state*. On the other hand, suppose you have a friend in a laboratory that produces a quantum system in some specific state, but you don't know what state it is in. You know that there is a probabilty P_n that your friend has made state $|\psi_n\rangle$. From your perspective, the system is in a *mixed state*, which is described by the classical probability of each quantum state. We can describe this mixed state by a *density operator* (also known as the *density matrix*)

   $$\rho \equiv \sum_n P_n \, |\psi_n\rangle\langle\psi_n| \ , \tag{1}$$

   which represents your ignorance of the true quantum state of the system. A density operator can be used to represent a thermal state of a quantum system (with probabilities given by the Boltzmann factor). A pure state is represented by a density operator where $P_n = 1$ for some specific state $|\psi_n\rangle$ with all other $P_n = 0$.

   (a) Suppose the system is the spin of an electron, and there is a 50% probability each that your friend has produced the spin either up along z or up along x. Write the density matrix first in terms of the states $|\uparrow_z\rangle, |\uparrow_x\rangle$ and then the basis states $|\uparrow_z\rangle, |\downarrow_z\rangle$. *Hint:* the S_x eigenstates are given in §4.4 of Griffiths.

   (b) Another way mixed states can arise is through entanglement. For two particles in the pure state $|\psi\rangle_{1,2}$, define the two-particle density operator as usual. Then the density operator for the first particle can be defined as

   $$\rho_1 = \sum_i {}_2\langle e_i|\rho|e_i\rangle_2 \ , \tag{2}$$

   where $|e_i\rangle$ is a basis of single-particle states. Suppose two electrons are in the $|s=0\rangle_{1,2}$ total spin state. Find the density operator of the first electron.

2. **The von Neumann Entropy**

   In the class notes (and reading), we briefly mentioned that quantum information should depend only on the state of the system and not on the measurements performed but should still reproduce the Shannon entropy. In this problem, we will briefly explore the quantum *von Neumann entropy*.

   The von Neumann entropy is defined as

   $$S \equiv -\sum_i \langle e_i| \left(\rho \log_2 \rho\right) |e_i\rangle \ , \tag{3}$$

   where the sum is over a complete basis of states $|e_i\rangle$. We can understand this by writing $\rho = 1 + (\rho - 1)$ and treating $\log_2 \rho$ as a power series in $(\rho - 1)$.

   (a) For a pure state $|\psi\rangle$, the density operator is $\rho = |\psi\rangle\langle\psi|$. Show that the von Neumann entropy (3) is zero for a pure state. *Hint:* Show that a pure state $|\psi\rangle$ has $(\rho - 1)|\psi\rangle = 0$ and define the logarithm as a power series of $\rho - 1$.

(b) Next, consider that an electron might be prepared with probability p spin up along z (that is, state $|\uparrow_z\rangle$) and probability $1-p$ spin down along z (in $|\downarrow_z\rangle$). Show that the von Neumann entropy is equal to the Shannon entropy $-p\log_2 p - (1-p)\log_2(1-p)$ for this system.

This means that

$$S = \langle\uparrow|\rho\log_2\rho|\uparrow\rangle + \langle\downarrow|\rho\log_2\rho|\downarrow\rangle = p\log_2 p + (1-p)\log_2(1-p)\ .$$

3. **White Dwarfs** *based on Griffiths 5.35*

   POSTPONED ONE ASSIGNMENT