PHYS-4601 Homework 13 Due 21 Jan 2016

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Center of Mass Frame and Reduced Mass**

   In class, we treat the hydrogen atom as if it is an electron moving around a stationary proton. Of course, that can't be, since it violates conservation of momentum. What happens, of course, is that the proton hardly moves in the center of mass rest frame. However, it turns out that we can always describe a system of two particles in terms of a single particle. In this problem, consider two particles of masses m_1 and m_2.

   (a) In quantum mechanics, the kinetic energy is given by a Laplacian operator. Consider the 1D case for simplicity. Then the kinetic Hamiltonian is

   $$H = -\frac{\hbar^2}{2m_1}\frac{d^2}{dx_1^2} - \frac{\hbar^2}{2m_2}\frac{d^2}{dx_2^2} \ , \tag{1}$$

   where x_1 is the first particle's position and x_2 is the second particle's position. Show that this kinetic Hamiltonian can be written as

   $$H = -\frac{\hbar^2}{2M}\frac{d^2}{dX^2} - \frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} \ , \tag{2}$$

   where $M = m_1 + m_2$ is the total mass, $\mu = m_1 m_2/M$ is the reduced mass, $X = (m_1 x_1 + m_2 x_2)/M$ is the center of mass position, and $x = x_1 - x_2$ is the relative position.

   The proof is essentially the same for the 3D Laplacian, and we then set the center of mass momentum to zero by choice of reference frame. Therefore, when we study the hydrogen atom, we are really using the reduced mass of the electron, which is nearly the electron mass because the proton is so much heavier than the electron.

   (b) Imagine an atom made of an electron and a positron (anti-electron); these are called positronium. Positronium atoms are exactly like hydrogen atoms (in terms of energy eigenvalues) except the proton mass is replaced by the positron mass (which is equal to the electron mass). Find the ground state energy and Bohr radius of positronium.

2. **Comparing Expectation Values**

   (a) *based on Griffiths 4.13* Find $\langle r^2 \rangle$ for the ground state of a hydrogen-like atom (a single electron moving in a central Coulomb potential) in terms of the Bohr radius. Find the ratio of this result between hydrogen to that for a helium ion, which has a single electron orbiting a nucleus of charge $+2e$. (In other words, find $\langle r^2 \rangle_H / \langle r^2 \rangle_{He^+}$.) What does this mean about the comparative "size" of these two atoms?

   (b) Now find the ratio of $\langle r^2 \rangle$ for the $n = 2, \ell = 1, m = 0$ state of hydrogen to the $n = 2, \ell = 0, m = 0$ state.

   (c) Finally, find the ratio of $\langle z^2 \rangle$ for the $n = 2, \ell = 1, m = 0$ state of hydrogen to the $n = 2, \ell = 0, m = 0$ state. *Hint:* You can find $\langle z^2 \rangle$ for the $n = 2, \ell = 0, m = 0$ state by using symmetry arguments and your work from part (b).

3. **Multiple Particle Wavefunctions** *based on a problem in Ohanian*

   Consider two free spin 1/2 particles, which have single particle states $|\psi_1\rangle = |\vec{p}_1\rangle|+\rangle$ and $|\psi_2\rangle = |\vec{p}_2\rangle|-\rangle$. These states are factorized into spatial states (in this case, momentum eigenstates) and spin states (eigenstates of S_z).

   (a) Write the two-particle wavefunction if the two particles are distinguishable (say, particle 1 is a proton and particle 2 is an electron).

   (b) Now suppose that both particles are electrons, so they are indistinguishable. Write the two-particle state which is an eigenfunction of the total spin operator $\vec{S}^2$ with eigenvalue given by $s = 0$.

   (c) Keeping indistinguishable electrons, now write the two-particle wavefunction for with total spin eigenvalues $s = 1$, $m = 0$.

   (d) Finally, consider the case where the particles are indistinguishable but instead have spin 0, so there is no spin part of their states. Write the allowed two-particle state.