PHYS-4601 Homework 11 Due 7 Jan 2016

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Raising and Lowering**

   (a) *More or less Griffiths 4.18* Using the relation for $L_\pm L_\mp$ given in class and the text, show that
   $$L_\pm|\ell,m\rangle = \hbar\sqrt{(\ell\mp m)(\ell\pm m+1)}|\ell,m\pm 1\rangle \ . \tag{1}$$

   (b) In a vector/matrix representation of the $\ell=1$ states where
   $$|1,1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix} \ , \quad |1,0\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix} \ , \quad |1,-1\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix} \ , \tag{2}$$
   use (1) to find matrix representations of $L_\pm$ and then L_x and L_y.

   (c) *Griffiths 4.22(b)* Use $L_+\cdot Y_\ell^\ell = 0$ and $L_z\cdot Y_\ell^\ell = \ell\hbar Y_\ell^\ell$ to determine $Y_\ell^\ell(\theta,\phi)$ up to overall normalization.

2. **Distinguishing Spins**

   Consider two beams of electrons. In beam 1, half of the electrons have spin up along z and half have spin down along z. In beam 2, all the electrons are in spin state $|\psi\rangle = (|+\rangle + i|-\rangle)/\sqrt{2}$.

   (a) As a preliminary calculation, find the eigenstates of S_y in the $|+\rangle, |-\rangle$ basis (note that the eigenstates of S_x are given in Griffiths equation [4.151]). *Hint:* Write $S_y = (\hbar/2)\sigma_y$ and then convert the eigenvectors back to Dirac notation at the end.

   (b) If you measure the electrons' S_z spins only, can you distinguish the two beams? Explain.

   (c) If you measure the electrons' S_y spins only, can you distinguish the two beams? Explain.

3. **Rotations** *parts of Griffiths 4.56*

   (a) Argue that $\exp[i\varphi L_z/\hbar]$ is a rotation around the z axis by showing that
   $$e^{i\varphi L_z/\hbar}\cdot\psi(\phi) = \psi(\phi+\varphi) \tag{3}$$
   for any angular wavefunction $\psi(\phi)$ that can be written as a Taylor series around ϕ. *Hint*: It will help to use the identification that $L_z = -i\hbar\partial/\partial\phi$.

   As a result, the angular momentum operators are called the *generators* of rotations. In general, $\hat{n}\cdot\vec{L}/\hbar$ generates rotations around the unit vector $\hat{n}$. Furthermore, the rotations of spinors are generated by the spin angular momentum operators.

   (b) What is the 2×2 matrix corresponding to a rotation of 2π around the z axis for spin 1/2? How does it compare to what you expected?

   (c) Construct the 2×2 matrix corresponding to a rotation of π around the x axis for spin 1/2. Show that it takes the S_z eigenstate $|+\rangle$ into $|-\rangle$.

4. **MRI Physics** *Inspired by a question by Horbatsch*

Consider a spin-1/2 proton with gyromagnetic ratio γ in the presence of a magnetic field

$$\vec{B} = B_0\hat{z} + B_1\cos(\omega t)\hat{x} - B_1\sin(\omega t)\hat{y} \tag{4}$$

at its fixed position. This is a magnetic field with a fixed z component and another component rotating in the x, y plane.

(a) Write the Hamiltonian in terms of spin operators and then as a matrix in the eigenbasis of the S_z operator.

(b) It is actually possible to find the full time-dependent state for this system. If the spin is up at $t = 0$, the solution of the time-dependent Schrödinger equation is

$$\begin{aligned} \langle +|\Psi(t)\rangle &= e^{i\omega t/2}\left[\cos(\alpha t/2) - i\frac{(\omega-\gamma B_0)}{\alpha}\sin(\alpha t/2)\right] \\ \langle -|\Psi(t)\rangle &= ie^{-i\omega t/2}\frac{\gamma B_1}{\alpha}\sin(\alpha t/2) \end{aligned} \tag{5}$$

with $\alpha = \sqrt{\gamma^2 B_1^2 + (\omega - \gamma B_0)^2}$. Use Maple to verify that (5) solves the Schrödinger equation. Input the Schrödinger equation and initial conditions as a list of equations and then the solution above as another list. Then use the `odetest` function in Maple to check that (5) solves the time-dependent Schrödinger equation. Include a copy of your Maple code. You may want to use the Maple help to learn how to use `odetest`.

(c) Use (5) to find the transition probability from spin up ($|+\rangle$) to spin down ($|-\rangle$). Find the conditions that this probability is one.

5. **Quantization of Electric Charge** *physics literacy*

In our usual studies of electromagnetism, we assume that electric monopole charges exist, but magnetic monopoles do not. In 1931, Paul Dirac (of Dirac notation fame) showed that the existence of a single magnetic monopole (anywhere in the universe) implies the quantization of electric charge. Since, of course, we observe that electric charge is quantized as in the Millikan oil drop experiment, this is intriguing. Dirac's argument, which you will reproduce here, uses only the combination of quantum mechanics and electromagnetic theory that you learned on a previous assignment.

Hint: In this problem, we will use functions of spherical polar coordinates and vectors as functions of spherical coordinates with components along the $\hat{r}, \hat{\theta}, \hat{\phi}$ directions. The gradient, divergence, and curl are (see the Griffiths EM Theory book, for example)

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}\ , \tag{6}$$

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\ , \tag{7}$$

$$\vec{\nabla}\times\vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi} \tag{8}$$

(a) Just like the electric field for a charge e is $\vec{E} = (e/r^2)\hat{r}$ (in appropriate units), the magnetic field for a monopole of magnetic charge q should be $\vec{B} = (q/r^2)\hat{r}$. However, one of

Maxwell's equations is $\vec{\nabla} \cdot \vec{B} = 0$. Show that the monopole field satisfies this Maxwell equation everywhere except possibly the origin, where the field is singular. This may seem paradoxical because the magnetic flux through a spherical Gaussian surface centered at the origin is non-zero, but really this is saying that the magnetic charge is a delta function at the origin.

(b) We recall that the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ in terms of the vector potential $\vec{A}$. Show that the two vector potentials

$$\vec{A}_N = \frac{q(1-\cos\theta)}{r\sin\theta}\hat{\phi} \quad \text{and} \quad \vec{A}_S = -\frac{q(1+\cos\theta)}{r\sin\theta}\hat{\phi} \tag{9}$$

both give the magnetic field expected for a magnetic monopole of magnetic charge q.

(c) Suppose we want to use $\vec{A}_N$ to describe $\vec{B}$. Unfortunately, it blows up at the south pole ($\theta = \pi$), so we are not allowed to use it there. Similarly, $\vec{A}_S$ blows up at the north pole ($\theta = 0$). Fortunately, we are allowed to use $\vec{A}_N$ for $\theta < \pi$ and $\vec{A}_S$ for $\theta > 0$ as long as the two vector potentials are related by a gauge transformation on the overlap region $0 < \theta < \pi$. Show that $\vec{A}_N = \vec{A}_S + \vec{\nabla}\Lambda$ for $\Lambda = 2q\phi$ in the overlap region.

(d) Now consider any charged particle of charge e moving around a magnetic monopole. Wherever we use $\vec{A}_N$ to describe the vector potential, the electron has wavefunction $\psi_N(\vec{x})$. However, where we switch to using $\vec{A}_S$ using a gauge transformation, we also have to transform the electron wavefunction to $\psi_S = \psi_N \exp(ie\Lambda/\hbar)$, as discussed on our previous assignment. Suppose ψ_N is single-valued (ie, periodic with period 2π in ϕ). Show that ψ_S is also single-valued only if $eq = \hbar N/2$, where N is an integer. This proves electric charge is quantized.