PHYS-4601 Homework 1 Due 17 Sept 2015

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Dirac Notation on the Circle**

   Consider the Hilbert space of L^2 functions on the interval $0 \leq x \leq 2\pi R$ with periodic boundary conditions.

   (a) Show that the complex exponentials $|e_n\rangle \simeq e^{inx/R}/\sqrt{2\pi R}$ for n any integer form an orthonormal set. As it turns out, they make a complete orthonormal basis.

   Carry out the following calculations *without doing any integrals*.

   (b) Calculate the inner product of $|f\rangle \simeq f(x) = \cos^3(x/R)$ and $|g\rangle \simeq g(x) = \sin(3x/R)$.

   (c) Find the inner product of $|f\rangle$ from part (b) with $|h\rangle \simeq h(x) = \sin(3x/R + \theta)$.

   (d) $|f\rangle, |g\rangle, |h\rangle$ are not normalized. Find their norms.

2. **Dual Vectors and Change of Basis**

   Consider a 3-dimensional complex column vector space, which has the usual orthonormal basis

$$|e_1\rangle \simeq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ , \ |e_2\rangle \simeq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ , \ |e_3\rangle \simeq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ . \tag{1}$$

   In that basis, the vectors $|f_i\rangle$ $(i = 1, 2, 3)$ can be written as

$$|f_1\rangle \simeq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \ , \ |f_2\rangle \simeq \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \ , \ |f_3\rangle \simeq \frac{1}{\sqrt{6}} \begin{bmatrix} i \\ -i \\ -2i \end{bmatrix} \ . \tag{2}$$

   (a) Write the $|f_i\rangle$ as linear superpositions of the $|e_i\rangle$ basis vectors.

   (b) Show that the $|f_i\rangle$ are normalized and mutually orthogonal, so they form a complete orthonormal basis (distinct from the set of $|e_i\rangle$).

   (c) Write the associated dual vectors $\langle f_i|$ as row vectors in the $\{\langle e_i|\}$ basis.

   (d) Write the $|e_i\rangle$ vectors as linear superpositions of the $|f_i\rangle$. Use your result to do a change of basis for this Hilbert space by writing the $|e_i\rangle$ vectors as column vectors in the $\{|f_i\rangle\}$ basis. *Hint:* You can solve a system of linear equations or use a similarity transformation, but it is much easier if you use inner products as discussed in the notes.

3. **Polynomial Vector Spaces** *inspired by Griffiths A.2*

   Consider the set of all polynomials up to order N in x with complex coefficients. This problem introduces other formulations of vector spaces and inner products.

   (a) Give a brief argument (not necessarily a detailed proof) that these form a vector space. List a convenient basis for this vector space. What is the dimensionality?

(b) The basis you listed for part (a) may or may not be orthonormal with respect to any inner product. Consider the inner product

$$\langle f|g\rangle = \int_0^\infty dx\, e^{-x} f^*(x) g(x) \text{ for } \langle x|f\rangle = f(x), \text{ etc} \tag{3}$$

and the Laguerre polynomials $L_n(x)$ for $0 \le x < \infty$ defined by

$$L_n(x) = e^x \left(\frac{d}{dx}\right)^n \left(e^{-x} x^n\right) \ . \tag{4}$$

The polynomial L_n is nth order (meaning the highest power of x in L_n is x^n). Show that L_n and L_m are orthogonal for $m \neq n$ and that L_0 and L_1 are normalized to unity. In fact, the Laguerre polynomials for $0 \le n \le N$ form an orthonormal basis for the vector space described in part (a).

4. **Superposition of States**

Suppose $|\psi\rangle$ and $|\phi\rangle$ are two normalized state vectors, and so is $|\alpha\rangle = A(3|\psi\rangle + 4|\phi\rangle)$.

(a) Find the normalization constant A in the case that

i. $\langle\psi|\phi\rangle = 0$.

ii. $\langle\psi|\phi\rangle = i$.

iii. $\langle\psi|\phi\rangle = e^{i\pi/6}$.

(b) In the case $\langle\psi|\phi\rangle = i$, use the *Gram-Schmidt procedure* described in Griffiths problem A.4 to find the part of $|\alpha\rangle$ orthogonal to $|\psi\rangle$. Verify that it is orthogonal by taking the inner product.