PHYS-3301 Winter Homework 2 Due 20 Jan 2016

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

Helpful Formulae: The following may be useful:

$$\int_0^\infty dx\, x^n e^{-x/a} = n!\, a^{n+1} \quad (a>0,\ n=0,1,2,\cdots) \tag{1}$$

$$\int_{-\infty}^\infty \frac{dk}{2\pi}\, e^{ik(x-x')} = \delta(x-x') \tag{2}$$

$$\int_{-\infty}^\infty dx\, x^{2n} e^{-b^2x^2} = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{1}{2b}\right)^{2n+1} \quad (\mathrm{Re}b>0)\ . \tag{3}$$

$$\tag{4}$$

1. **Position and Momentum Operators in Momentum Space**

   For a particle moving on the real line ($-\infty < x < \infty$) with wavefunction $\psi(x)$, the "momentum space wavefunction" is

   $$\tilde{\psi}(p) = \int_{-\infty}^\infty dx\, \psi_p(x)^*\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^\infty dx\, e^{-ipx/\hbar}\psi(x)\ . \tag{5}$$

   (a) If the position operator acts on a wavefunction $\psi(x)$, it returns the wavefunction $x\psi(x)$. Show that the momentum space wavefunction corresponding to this is $i\hbar d\tilde{\psi}/dp$.

   (b) If the momentum operator acts on a wavefunction $\psi(x)$, it returns the wavefunction $-i\hbar d\psi/dx$. Show that the momentum space wavefunction corresponding to this is $p\tilde{\psi}(p)$.

2. **Normalization and Expectation Values**

   In this problem, $\tilde{\psi}$ will still represent the momentum space wavefunction as in equation (5).

   (a) Show that normalization of the momentum space wavefunction implies normalization of the position space wavefunction. That is, show

   $$\int_{-\infty}^\infty dp\, |\tilde{\psi}(p)|^2 = 1 \ \Rightarrow\ \int_{-\infty}^\infty dx\, |\psi(x)|^2 = 1\ . \tag{6}$$

   Hint: Substitute two factors of $\tilde{\psi}$ from equation (5) using different dummy integration variables. Then use the definition of the delta function (2).

   (b) Suppose the wavefunction is a normalized Gaussian $\psi(x) = (2a/\pi)^{1/4}\exp(-ax^2)$. Find the momentum space wavefunction. *Hint:* Combine exponentials in equation (5) and complete the square in the exponent. Then shift integration variables to get a Gaussian integral.

   (c) The expectation value $\langle p^2\rangle$ can be calculated in two ways:

   $$\langle p^2\rangle = -\hbar^2\int_{-\infty}^\infty dx\, \psi^*(x)\frac{d^2\psi}{dx^2} \ \text{ or }\ \langle p^2\rangle = \int_{-\infty}^\infty dp\, p^2|\tilde{\psi}(p)|^2\ . \tag{7}$$

   Use your results from the previous part to show that these are equal for the Gaussian wavefunction.

3. **Spherical Harmonics**

You may want to consult table 6.1 on page 132 of Scherrer.

(a) Write the function $\cos(2\theta)$ in terms of spherical harmonics.

(b) Is $\cos(2\theta)$ an eigenfunction of L_z or $\vec{L}^2$ (or both)? Give the eigenvalue for any operator for which it is an eigenfunction.

(c) Find the probability that an electron in the $n = 3, \ell = 0, m = 0$ state of hydrogen is found in the first octant of space (that is, is located in the region with $x > 0, y > 0, z > 0$). Then find the probability if the electron is in the $n = 2, \ell = 1, m = 1$ state.