Relativistic Dynamics

4-Momentum

It should be clear that Newtonian definitions of momentum and energy won't work. We need something new.

— How can we guess at momentum and energy?

- We would like something that's a 4-vector, scalar, or even tensor. This will let us write conservation laws as manifestly covariant equations
- As we have seen, coordinate velocity $\vec{u} = d\vec{x}/dt$ is poorly suited to this
- What can we use to define a momentum
  + The 4-velocity U^μ
  + The mass m. This is the mass as measured in the particle's rest frame and is a scalar (b/c it refers to a frame specific for that particle). ← Lorentz inv't

    It's "bad form" to define a velocity-dependent mass.

→ Guess at a momentum 4-vector $p^\mu = m U^\mu$

- As with velocity, components are

$$p^0 = \gamma m c \quad , \quad \vec{p} = \gamma m \vec{u}$$

- Does this make sense? At low speeds

$$\vec{p} \simeq m\vec{u}\left(1 - \vec{u}^2/c^2\right)^{-1/2} \simeq m\vec{u}\left(1 + \frac{1}{2}\frac{\vec{u}^2}{c^2} + \cdots\right)$$

That indeed approximates the Newtonian value

- Momentum conservation says
  1) Take total initial momentum over all particles $P_i^\mu = \sum_j p_j^\mu$
  2) Take total final momenta of all particles $P_f^\mu = \sum_k p_k^\mu$
  3) These are equal $P_i^\mu = P_f^\mu$. This is Covariant.
  + The number of particles may change.
  + We have a new 4th momentum conservation $P_i^0 = P_f^0$ in addition to the regular momentum conservation
  + But mass need not be conserved.
- What is the 0^{th} component of momentum conservation?
  + Look at a single particle again for small $\vec{u}^2$:

$$p^0 = \gamma mc \simeq mc\left(1 + \frac{1}{2}\frac{\vec{u}^2}{c^2} + \dots\right) = \frac{1}{c}\left(mc^2 + \frac{1}{2}m\vec{u}^2 + \dots\right)$$

  This is apparently E/c, where the energy E includes a rest energy mc^2 and the Newtonian kinetic energy plus relativistic corrections.
  + Stated again: The time component of a particle's 4-momentum is its total relativistic energy.
  + Conservation of momentum includes conservation of energy.
- The text lays out experimental evidence:
  + Speed vs energy
  + Energy release due to change of mass (see below) in two examples

— Momentum, Energy, + Mass:

Let's settle on our definition $p^\mu = mU^\mu$ for 1 particle

- Like velocity, p^μ is timelike for a massive particle

$$p^2 = p_\mu p^\mu = m^2 U_\mu U^\mu = -m^2c^2 \leftarrow \text{very important}$$

+ The invariant square of 4-momentum is given by the mass
+ In terms of components, we see

$$p_\mu p^\mu = (-E/c)(E/c) + \vec{p}^{\,2} = \frac{1}{c^2}\left(-E^2 + \vec{p}^{\,2}c^2\right) = -m^2c^2$$

Rearrange

$$E^2 = m^2c^4 + \vec{p}^{\,2}c^2 \leftarrow \underline{\text{very}}\ \underline{\text{important}}$$

+ This is the origin of $E = mc^2$.

- What are the consequences for multiple particles?

+ Just total up free-particle momenta: $p_1^0 + p_2^0 + \cdots p_n^0 = E_{tot}/c = P^0_{tot}$ and $\vec{p}_1 + \cdots \vec{p}_n = \vec{P}_{tot}$. P^2_{tot} is a relativistic invariant of course

We define $P^2_{tot} = -M^2c^2$ called invariant mass

+ What about potential energy?

Momentum adds up the same way $\vec{P}_{tot} = \vec{p}_1 + \cdots + \vec{p}_n$

But total energy includes potential $P^0_{tot} = E_{tot}/c = p_1^0 + \cdots + p_n^0 + V/c$.

Can still define $P^2_{tot} = -M^2c^2$

+ The combined (total) energy-momentum including potential gives the mass of a bound state (... M ...

Look at overall rest frame $\vec{P}_{tot} = 0$. $M = \text{mass/kinetic} + \text{potential}$

Examples

1) Chemical energy: $m_p \sim 1\,\text{GeV}/c^2$, $m_e \sim 0.5\,\text{MeV}/c^2$

$KE + V = $ eV to maybe 100s of eV (13.6 eV for H ground state)

This is why classically mass looks conserved.

2) Nuclear energies: m_p to 100ish$\times m_p$
Binding energies ~ 0.1 to a few MeV ← still not too relativistic
Can release these energies by splitting (fission) or merging (fusion)

3) Particle Physics: Can release energy ~~completely~~ completely into kinetic energy, like electron/positron (antielectron) annihilation.

- Massless particles:
  + Take the limit $m \to 0$, so $p^2 = p_\mu p^\mu = 0$.
    For a normal particle $|\vec{u}|/c = |\vec{p}|/p^0 \to 1$.
    So these particles move at the speed of light.
    Examples: photons (light particles), gravitons,
  + Re-emphasize: massless particles have energy-momentum. The 4-vector is light-like.

— The Center of Momentum (CM) frame for Collisions

- Collisions usually simplify in the frame where $\vec{P}_{tot} = 0$. ← initial + final
  We call this the CM frame; it is the same as the nonrelativistic center of mass
  + Book gives it the name S^* and labels CM variables with $*$
  + In a particle decay (ie, 1 particle to many), it is initial rest frame.
    In 2 particle collision it is the head-on collision at equal momenta
  + Calculate in CM frame: $P_{tot}^2 = -(E^*/c)^2 + \vec{P}_{tot}^2 = -(E^*/c)^2$
    So P_{tot}^2 (a relativistic invt) tells you the CM frame energy.
- Other possibly interesting frames:
  + Lab frame: particle colliding with fixed target $\longrightarrow \bullet$
  + Rest frame of total cosmic fluid in early universe