

## Particle 4-Velocity

The idea is to write a 4-vector for velocity

- Notice that the proper time along the worldline of a particle is Lorentz invariant and spacetime position is a 4-vector.

• Specifically,  $d\tau = dt \sqrt{1 - \vec{u}^2/c^2}$  is Lorentz invariant.

• Also,  $dx^\mu = (cdt, d\vec{x})$  is a 4-vector

• Now consider  $U^\mu = dx^\mu/d\tau$  derivative wrt. proper time

The "denominator" is Lorentz invariant, "numerator" is 4-vector.

Therefore  $U^{\mu'} = \frac{d}{d\tau} (\Lambda^{\mu'}_{\nu} x^\nu) = \Lambda^{\mu'}_{\nu} \frac{dx^\nu}{d\tau} = \Lambda^{\mu'}_{\nu} U^\nu$

•  $U^\mu$  is a 4-vector. We call it 4-velocity.

- The components are interesting in terms of the normal velocity

$$U^0 = c \frac{dt}{d\tau} = \frac{c}{\sqrt{1 - \vec{u}^2/c^2}} = c \gamma(\vec{u})$$

$$\vec{U} = \frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau} \frac{d\vec{x}}{dt} = \gamma(\vec{u}) \vec{u}$$

Note that  $|\vec{U}| \rightarrow \infty$  as  $|\vec{u}| \rightarrow c$ .

• Inverting this relationship, the coordinate velocity is

$$\vec{u} = \vec{U} / U^0$$

- Comments + properties:

•  $U^0 > 0$  for any sensible particle traveling into the future

•  $U^\mu$  is always timelike and has square

$$U^2 = U_\mu U^\mu = -c^2 \gamma^2 + \gamma^2 \vec{u}^2 = -c^2 \gamma^2 (1 - \vec{u}^2/c^2) = -c^2$$

This is true for any 4-velocity of any particle

• This is really only defined for massive particles (ie, not photons)