Tensors

- We've defined 4-vectors by their transformation
  - A (contravariant) 4-vector a^μ has Lorentz transformation $a^{\mu'} = \Lambda^{\mu'}{}_\nu a^\nu$. In reverse, $a^\mu = \Lambda^\mu{}_{\nu'} a^{\nu'}$. These have upper indices

    In matrix form,

$$\left[\Lambda^{\mu'}{}_\nu\right] \left(\underset{\text{transformation}}{S \to S'}\right) = \left[\Lambda^\mu{}_{\nu'}\right]^{-1} \left(S' \to S \text{ transformation}\right)$$

  - But the metric allows us to define covariant (or dual) 4-vectors with lower indices. $a_\mu \equiv \eta_{\mu\nu} a^\nu$ (remember the sum)

    Component-by-component, $a_0 = -a^0$, $a_i = a^i$
  - What's the Lorentz transformation?

    By definition, $a_{\mu'} = \eta_{\mu'\nu'} a^{\nu'} = \Lambda^\alpha{}_{\mu'} \Lambda^\beta{}_{\nu'} \eta_{\alpha\beta} \Lambda^{\nu'}{}_\lambda a^\lambda$

    Because of the matrix inverse above, $\Lambda^\beta{}_{\nu'} \Lambda^{\nu'}{}_\lambda = \delta^\beta_\lambda$, so

$$a_{\mu'} = \Lambda^\alpha{}_{\mu'} \eta_{\alpha\beta} a^\beta = \Lambda^\nu{}_{\mu'} a_\nu$$

  - In other words, a lower index transforms by the inverse transformation. In summary

    $S \to S'$: $a^{\mu'} = \Lambda^{\mu'}{}_\nu a^\nu$ and $a_{\mu'} = \Lambda^\nu{}_{\mu'} a_\nu$

    $S' \to S$ $a^\mu = \Lambda^\mu{}_{\nu'} a^{\nu'}$ and $a_\mu = \Lambda^{\nu'}{}_\mu a_{\nu'}$
  - We've lowered indices using $\eta_{\mu\nu}$. We can raise indices with $\eta^{\mu\nu}$, the inverse metric.

$$\left[\eta^{\mu\nu}\right] = \left[\eta_{\mu\nu}\right]^{-1} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \quad \text{so} \quad a^\mu = \eta^{\mu\nu} a_\nu$$

- Contractions
  - We can write the scalar product in several ways

$$a \cdot b = \eta_{\mu\nu} a^\mu b^\nu = a_\nu b^\nu = a^\mu b_\mu = \eta^{\mu\nu} a_\mu b_\nu$$

+ An upper index summed with a lower index is <u>contracted</u>.
You can only contract identical upper + lower indices

+ Contracted indices lead to Lorentz invariants (like scalar products).

$$a_{\mu'}b^{\mu'} = a_\nu \left(\Lambda^\nu{}_{\mu'} \Lambda^{\mu'}{}_\lambda\right) b^\lambda = a_\nu \delta^\nu{}_\lambda b^\lambda = a_\nu b^\nu$$

Contracted indices "disappear" from Lorentz transformations (like contracted letters in a word)

<u>Tensors</u> are objects with multiple indices where each index transforms like a <u>separate</u> 4-vector (upper or lower) index

+ For example, $T_{\mu\nu}{}^\lambda$ is a tensor w/ 2 lower + 1 upper index, so

$$T_{\mu'\nu'}{}^{\lambda'} = \Lambda^\alpha{}_{\mu'} \Lambda^\beta{}_{\nu'} \Lambda^{\lambda'}{}_\gamma T_{\alpha\beta}{}^\gamma$$

+ The only tensor we have seen so far is the metric $\eta_{\mu\nu}$ (and the inverse $\eta^{\mu\nu}$). Recall $\eta_{\mu'\nu'} = \Lambda^\lambda{}_{\mu'} \Lambda^\rho{}_{\nu'} \eta_{\lambda\rho}$.
This is actually an invariant tensor $[\eta_{\mu'\nu'}] = [\eta_{\mu\nu}]$ as matrices

+ Another example is the <u>Levi-Civitá</u> <u>tensor</u> $\epsilon_{\mu\nu\lambda\rho}$
This is totally antisymmetric, so it $=0$ if any of μ, ν, λ, ρ are equal and is $=\pm 1$ otherwise depending on the permutation of the indices. That is, $\epsilon_{0123} = +1$, $\epsilon_{1023} = -1$, etc.
This is also invariant: $\epsilon_{\mu'\nu'\lambda'\rho'} = \epsilon_{\mu\nu\lambda\rho}$ component-by-component

+ In special relativity, electromagnetic fields form a tensor $F_{\mu\nu}$.
It is <u>not</u> invariant.

+ Tensor indices can be raised and lowered, like 4-vectors

$$T_\mu{}^{\nu\lambda} = T_{\mu\alpha}{}^\lambda \eta^{\nu\alpha}, \quad T^{\mu\nu}{}_\lambda = T_{\alpha\beta}{}^\gamma \eta^{\mu\alpha} \eta^{\nu\beta} \eta_{\lambda\gamma}$$

etc

+ You can also contract tensor indices like 4-vector indices

So

$$T_{\mu\nu}{}^{\lambda} a_\lambda = S^{(1)}_{\mu\nu} = \text{tensor w/ 2 lower indices}$$

$$T_{\mu\nu}{}^{\lambda} a^\nu = S^{(2)}{}_{\mu}{}^{\lambda} = \text{tensor w/ 1 lower + 1 upper index}$$

$$T_{\mu\nu}{}^{\lambda} a^\nu b_\lambda = S^{(3)}{}_{\mu} = \text{(covariant) 4-vector}$$

$$T_{\mu\nu}{}^{\lambda} a^\mu b^\nu = S^{(4)\lambda} = \text{(contravariant) 4-vector}$$

$$T_{\mu\nu}{}^{\lambda} a^\mu b^\nu d_\lambda = S^{(5)} = \text{scalar}$$

$$T_{\mu\nu}{}^{\lambda} T_{\alpha\beta}{}^{\gamma} \eta^{\mu\alpha} \eta^{\nu\beta} \eta_{\lambda\gamma} = \text{scalar, etc}$$

Covariance of Physical Equations

- As we've said, physical laws should have the same form in all inertial frames. All terms should be covariant (transform in the same way)
  + Raising / lowering indices are different ways to write the same object in one inertial frame. Contractions pair + sum indices in a Lorentz-invariant way. This means Lorentz transformations are determined only by uncontracted indices.
  + An equation is covariant iff all uncontracted indices are the same in each term.
  + Examples:

$$a^\mu f_{\mu\nu} g_\lambda = b_{\nu\lambda} \checkmark \qquad u^\alpha u_\beta = p^\mu \times \text{ unmatched indices}$$

$p^\mu + k^\mu = q^\mu$ ✓ $\qquad a^\nu T^{\mu\mu} = b^\nu$ ✗ can't contract 2 upper indices

$a^\mu b^\mu q_{\mu} p^\lambda = d^{\mu\lambda}$ ✗ what is contracted?

- We will see covariant eqns of various forms, but it is useful to work with eqns. of the form (scalar) = (scalar). Then you don't have to worry so much about changing frames.