Introduction to General Relativity $c=1$ units

The Metric

- In special relativity, we use the metric tensor $\eta_{\mu\nu}$
  - We remember that this has components $\eta_{00}=-1$, $\eta_{ij}=\delta_{ij}$, $\eta_{0i}=0$.
  - This is the "flat spacetime" or "Minkowski" metric
- But, with gravity included, the metric becomes a dynamical variable.
  - We need to replace $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$, a function of position
  - Like $\eta_{\mu\nu}$, $g_{\mu\nu}$ should be symmetric $g_{\mu\nu}=g_{\nu\mu}$
  - The invariant interval becomes $ds^2 = g_{\mu\nu}(x)\,dx^\mu dx^\nu$, and we can use any kind of coordinates we want. Examples:
    + Schwarzschild metric (metric around a spherically symmetric mass M) $\quad ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$
    + Friedman-Robertson-Walker metric (of universe)

      $$ds^2 = -dt^2 + a(t)^2(dx^2+dy^2+dz^2) \leftarrow \text{"flat" version}$$
  - The metric controls how matter moves. Matter determines the metric through the Einstein equations
- Conservation Laws
  - Just like momentum is not conserved when there is a potential energy function that depends on position, momentum-energy is not conserved for a particle when the metric depends on spacetime
  - On the other hand, if the metric does not depend on a coordinate (like the Schwarzschild metric is independent of time), that component of a particle's (lowered) 4-momentum is conserved.
  - Stated without proof: If $\frac{\partial}{\partial x^\mu} g_{\nu\lambda}(x) = 0$ for all elements of the metric, P_μ is conserved along the particle's path.

- Schwarzschild Metric
  - Birkhoff's theorem: the unique solution of Einstein eqns with spherical symmetry is the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{\left(1-\frac{2GM}{r}\right)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

    - This includes outside the earth, sun, etc
    - You will notice a similarity to gravitational potential energy, so it relates to Newtonian gravity
  - What does it mean?
    - Time dilation: Remember that $d\tau^2 = -ds^2$.
      + For an object at rest, $d\tau = \sqrt{1-2GM/r}\, dt$
      + Time t = proper time τ for object at $r=\infty$
      + Proper time passes more slowly closer to the center (smaller r)
      + GPS units have to account for this effect
    - Gravitational red shift
      + From our conservation laws, the wavevector component k_0 is conserved for any light ray
      + Suppose we have an observer at rest at radius r. Remember 2 things: $U_\mu U^\mu = -1$, $U^\mu k_\mu = -\omega_{observed}$
      + Then $U^i = 0$, and the 1st means $U^0 = \left(1 - \frac{2GM}{r}\right)^{-1/2}$. That means the observed frequency is $\omega \propto 1/\sqrt{1-\frac{2GM}{r}}$.
      + Frequency of a light ray decreases as r increases (photon is spending energy to "climb the hill")
      + Measured by Pound + Rebka in 1960
    - Black holes
      + At $r \to 2GM$, $g_{00} \to 0$, $g_{rr} \to \infty$. Very strange!
      + Remember that light rays have $ds^2 = 0$.

That implies $dr/dt = 1 - 2GM/r \to 0$ as $r \to 2GM$.
Light doesn't move!

+ $r = 2GM$ is the Schwarzschild radius and it's the horizon of the black hole. Nothing can escape the horizon.

- FRW metric
  - The universe on average is the same everywhere, and that was especially true early on. The FRW metric describes this:
  $$ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2)$$
    - This is the version with flat space, in agreement w/observation
    - $a(t)$ = scale factor, which increases with time. It shows the expansion of the universe
      + The distance between galaxies is $\Delta s = a(t) D$, where D = distance in coordinates. This is an increase in space, not the motion of anything
      + Sometimes it is written as a velocity by differentiating $v = \dot{\Delta s} = \dot{a} D = \left(\frac{\dot{a}}{a}\right) \Delta s \equiv H \Delta s$
  - Cosmological redshift
    - Again, consider observers at rest at times t_E and t_R. They both have $U^i = 0$ and $U_\mu U^\mu = -1 \Rightarrow U^0 = 1$
    - The metric is independent of x, y, z. Consider a light wave moving in the x direction, so $k_y = k_z = 0$. By conservation, $k_x = \text{const}$.
    - But we know $k_\mu k^\mu = 0$. This is $-(k_0^0)^2 + \frac{1}{a^2}(k_x)^2 = 0$
    $$\Rightarrow k_0 \propto 1/a$$
    - Therefore, observed and emitted frequencies are
    $$\omega_E = -U_E^\mu k_\mu \propto 1/a(t_E) \quad \text{and} \quad \omega_R = -U_R^\mu k_\mu \propto 1/a(t_R)$$
    - So light is redshifted $\omega_R/\omega_E = a(t_E)/a(t_R) < 1$ for $t_R > t_E$
    - Sometimes, people say this is a Doppler effect $\delta\lambda/\lambda = v/c$. But cosmological redshift can be >1. This is not motion faster than light!