PHYS-3301 Homework 7 Due 29 Oct 2014

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Some Short Practice Calculations**

   Calculate the following quantities. You should get a number for each answer.

   (a) $\eta_{\mu\nu}\eta^{\mu\nu}$

   (b) $\eta^{\mu\nu}\eta^{\lambda\rho}\epsilon_{\mu\nu\lambda\rho}$

   (c) $\epsilon_{\mu\nu\lambda\rho}\epsilon^{\mu\nu\lambda\rho}$

   Based on a problem by Sean Carroll In the next two calculations, define the tensor and vector

$$\begin{bmatrix} & & \\ & X^{\mu\nu} & \\ & & \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{bmatrix} \ , \quad V^\mu = (-1, 2, 0, -2) \tag{1}$$

   in some inertial frame S. Then calculate the following:

   (d) $X^\mu{}_\mu$

   (e) $X^{\mu\nu}V_\mu V_\nu$

2. **Derivatives Have Lowered Indices**

   As discussed in the class notes, 4-vectors with raised or lowered indices have the following Lorentz transformations:

$$a^{\mu'} = \Lambda^{\mu'}{}_\nu a^\nu \quad \text{and} \quad a_{\mu'} = \bar{\Lambda}_{\mu'}{}^\nu a_\nu \ , \tag{2}$$

   where Λ is the usual Lorentz transformation matrix and $\bar{\Lambda}^T = \Lambda^{-1}$ as a matrix.

   (a) Show that the matrix relationship between $\bar{\Lambda}$ and Λ may be written as $\bar{\Lambda}_{\mu'}{}^\rho \Lambda^{\nu'}{}_\rho = \delta^{\nu'}_{\mu'}$ and $\bar{\Lambda}_{\rho'}{}^\mu \Lambda^{\rho'}{}_\nu = \delta^\mu_\nu$, where $\delta^{\nu'}_{\mu'}$ and δ^μ_ν are Kronecker delta symbols.

   (b) Using the fact that the spacetime position x^μ is a 4-vector, find the partial derivatives $\partial x^\mu/\partial x^{\nu'}$ and $\partial x^{\mu'}/\partial x^\nu$ in terms of $\Lambda^{\mu'}{}_\nu$ and $\bar{\Lambda}_{\mu'}{}^\nu$. *Hint:* For two positions as measured in the same frame, $\partial x^\mu/\partial x^\nu = \delta^\mu_\nu$ (think about why).

   (c) If f is a Lorentz invariant function (meaning its value at a fixed spacetime point is the same in any frame — like the temperature), use the chain rule to show that

$$\frac{\partial f}{\partial x^{\mu'}} = \bar{\Lambda}_{\mu'}{}^\nu \frac{\partial f}{\partial x^\nu} \ . \tag{3}$$

   In other words, you are showing that a partial derivative has the same transformation as a 4-vector with a lowered index. As a result, people will usually write $\partial_\mu f \equiv \partial f/\partial x^\mu$.

3. **The Relativistic Electromagnetic Field**

   We won't prove it, but the electric and magnetic fields can be written as a relativistic tensor with two indices $F^{\mu\nu}$. This tensor is *antisymmetric*, meaning $F^{\nu\mu} = -F^{\mu\nu}$. The independent components are (here, $i = 1, 2, 3$ is a space index)

$$F^{0i} = E^i \ , \quad F^{12} = B^3 \ , \quad F^{13} = -B^2 \ , \quad F^{23} = B^1 \ . \tag{4}$$

Since $F^{\mu\nu}$ is antisymmetric, the diagonal components $F^{00} = F^{11} = F^{22} = F^{33} = 0$. (We have chosen a convenient system of units where the electric and magnetic field have the same dimension.)

(a) Consider two frames S and S' in standard configuration with each other. Show that

$$E^{3'} = \gamma\left(E^3 + \frac{v}{c}B^2\right) \quad \text{and} \quad B^{3'} = \gamma\left(B^3 - \frac{v}{c}E^2\right) \ . \tag{5}$$

Hint: Remember that the Lorentz transformation of a tensor transforms each index independently:

$$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\alpha}\Lambda^{\nu'}{}_{\beta}F^{\alpha\beta} \ . \tag{6}$$

(b) Calculate $F_{\mu\nu}F^{\mu\nu}$ and argue that $\vec{E}^2 - \vec{B}^2$ is a Lorentz invariant quantity.

4. **Energy-Momentum Tensor**

The energy-momentum tensor $T^{\mu\nu}$ describes the energy and momentum densities of a fluid. The components are defined as $T^{00} = \rho$, the energy density; $T^{0i} = T^{i0} = \mathcal{P}^i$, the density of the ith component of momentum; $T^{ij} = \sigma^{ij}$ for $i \neq j$ describes the shear stress; and T^{ij} with $i = j$ is the pressure P (we assume that the pressure is the same in all directions).

(a) Calculate $\eta_{\mu\nu}T^{\mu\nu}$.

(b) In the fluid's rest frame S, $\mathcal{P}^i = 0$ and $\sigma^{ij} = 0$. What is the momentum density in a frame S' moving at speed v in the $+x$ direction with respect to S if the fluid is radiation with $P = \rho/3$?