PHYS-3301 Homework 6 Due 22 Oct 2014

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Boosts and Rotations**

   In matrix form, we can define the boost Λ_{tx} along x and the rotation Λ_{xy} in the xy plane (around the z axis) as follows:

$$\Lambda_{tx}(\phi) = \begin{bmatrix} \cosh\phi & -\sinh\phi & & \\ -\sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} , \quad \Lambda_{xy}(\theta) = \begin{bmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{bmatrix} . \tag{1}$$

   Empty elements in the matrices above are zero.

   (a) In matrix form, the metric $\eta_{\mu\nu}$ is

$$\eta = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} . \tag{2}$$

   Show that both rotation and boost in (1) satisfy the condition $\eta_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta}$, which is $\eta = \Lambda^T \eta \Lambda$ in matrix notation.

   (b) Consider two successive boosts along x, $\Lambda_{tx}(\phi_1)$ and $\Lambda_{tx}(\phi_2)$. Show that these multiply to give a third boost $\Lambda_{tx}(\phi_3)$ and find ϕ_3. Using the relationship $v/c = \tanh\phi$ between velocity and rapidity ϕ, reproduce the velocity addition rule. *Hint:* You will need the angle-addition rules for hyperbolic trig functions.

   (c) First, write down the Lorentz transformation matrix $\Lambda_{ty}(\phi)$ corresponding to a boost along the y direction by permuting axes. Then show that you can get a boost along y by rotating axes, boosting along x, then rotating back by proving that $\Lambda_{ty}(\phi) = \Lambda_{xy}(-\pi/2)\Lambda_{tx}(\phi)\Lambda_{xy}(\pi/2)$.

2. **4-Vectors and Changing Frames**

   For parts (a,b) define a 4-vector U^μ to have components $U^0 = \cosh\phi$, $U^1 = \sinh\phi$, $U^2 = U^3 = 0$ in some reference frame S.

   (a) Now boost to reference frame S' which has velocity $v = c\tanh\theta$ along the x direction relative to S. What are the components of U^μ in S'? Write your answer in terms of the "angle" $\phi - \theta$.

   (b) If $\phi - \theta$ is itself a rapidity, write the associated velocity in terms of $\tanh\theta$ and $\tanh\phi$.

   In parts (c)-(e), a^μ and b^μ are both timelike.

   (c) If a^μ is timelike, show that there exists an inertial frame S where the only nonzero component is a^0 (that is, the spatial part $\vec{a}$ is zero).

   (d) Without using explicit Lorentz transformations, show that $|a^0|$ is minimized in the frame S defined in part (c).

(e) *from the text by Hartle* Show that $a \cdot b = -\sqrt{a^2 b^2}\gamma$, where γ is the relativistic γ factor for the Lorentz transformation between the frame S where the spatial part of a^μ is zero and the frame S' where the spatial part of b^μ is zero.

(f) Some component of the 4-vector k^μ is zero in every inertial reference frame. Show that k^μ is the zero vector.

3. **Some Scalar Products**

In some frame, the components of two 4-vectors are

$$a^\mu = (2, 0, 0, 1) \text{ and } b^\mu = (5, 4, 3, 0) \ . \tag{3}$$

inspired by a problem in Hartle

(a) Find a^2, b^2, and $a \cdot b$.

(b) Does there exist another inertial frame in which the components of a^μ are $(1, 0, 0, 1)$? What about b^μ? Explain your reasoning.

Now consider lightlike 4-vectors a^μ and b^μ.

(c) If a^μ and b^μ are orthogonal ($a \cdot b = 0$), prove that they are parallel ($a^\mu \propto b^\mu$).

(d) Is the 4-vector $a^\mu + b^\mu$ spacelike, timelike, or lightlike?