Quantum Mechanics I PHYS-3301 December Test

Dr. Andrew Frey

11 Dec 2014, 9AM-12PM, 3M64

Instructions:

- Do not turn over until instructed. You will have 3 hours to complete this test.
- No electronic devices or hardcopy notes are allowed.
- INSTRUCTIONS ABOUT THE QUESTIONS WILL GO HERE.
- **Only the lined pages of your exam book will be graded. Use the blank pages for scratch work only.**

Useful Concepts & Formulae:

- Notation and nomenclature
  - Frames S and S' are in standard configuration if their spacetime origins $(t, \vec{x}) = 0$ and $(t', \vec{x}') = 0$ overlap, their spatial axes point in the same directions, and their relative velocity is along x.
  - If you need the speed of light in a calculation, use $c = 3 \times 10^8$ m/s = 1 lightsecond/second as appropriate.
  - The CM frame is the frame in which the total spatial momentum is zero.
  - Einstein summation convention: repeated indices are summed.
- Galilean Relativity/Newtonian Mechanics
  - Galilean boost $\vec{x}' = \vec{x} - \vec{v}t$, $\vec{u}' = \vec{u} - \vec{v}$, $\vec{p}' = \vec{p} - m\vec{v}$, $k' = k - \vec{p} \cdot \vec{v} + (1/2)mv^2$
  - Kinetic energy for many particles $K = K_{int} + (1/2)MV^2$

$$M = \sum m_i \ , \quad \vec{V} = \frac{1}{M} \sum m_i \vec{u}_i$$

  - For two particles $K_{int} = (1/2)\mu u^2$ for relative velocity $\vec{u}$ and reduced mass $\mu = m_1 m_2 / M$
- 4-vectors and Lorentz transformations
  - The position 4-vector is x^μ with $x^0 = ct$.
  - The metric $\eta_{\mu\nu}$ can be written as a diagonal matrix with diagonal elements $-1, 1, 1, 1$, and the invariant interval is $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + d\vec{x}^2$.
  - The Lorentz boost transformations (in standard configuration) are

$$t' = \gamma(t - vx/c^2) \ , \ x' = \gamma(x - vt) \ , \quad y' = y \ , \quad z' = z \ , \quad \gamma = 1/\sqrt{1 - v^2/c^2} \ .$$

    They can be written as $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$
  - Lowered indices $a_\mu = \eta_{\mu\nu} a^\nu$ (both in frame S)
  - Relativistic dot product $a \cdot b = \eta_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu = -a^0 b^0 + \vec{a} \cdot \vec{b}$
  - Tensor transformation $T_{\mu' \cdots}{}^{\nu' \cdots} = (\bar{\Lambda}_{\mu'}{}^\alpha \cdots)(\Lambda^{\nu'}{}_\beta \cdots) T_{\alpha \cdots}{}^{\beta \cdots}$ with $\bar{\Lambda} = (\Lambda^T)^{-1}$

- Velocities and Momenta
  - For a normal velocity $\vec{u} = d\vec{x}/dt$, the Lorentz transformation between two frames in standard configuration with relative velocity v is

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2} \ , \quad u'_{y,z} = \frac{u_{y,z}}{\gamma(v)(1 - vu_x/c^2)} \ .$$

  - The 4-velocity of a particle is $U^\mu = dx^\mu/d\tau$, where τ is the proper time along the particle's worldline. $U^0 = \gamma c$, $\vec{U} = \gamma d\vec{x}/dt$, so $d\vec{x}/dt = c(\vec{U}/U^0)$.
  - 4-momentum is $p^\mu = mU^\mu$. Energy $E = cp^0$ and momentum is the spatial part $\vec{p}$.
  - $U_\mu U^\mu = -c^2$ and $p_\mu p^\mu = -m^2c^2$ for a normal massive particle.
- The Doppler effect, in terms of the rest frame of the receiver, is

$$\frac{\omega_R}{\omega_E} = \frac{\sqrt{1 - u_E^2/c^2}}{1 - \hat{k} \cdot \vec{u}_E/c} \ ,$$

  where $\hat{k}$ is the direction of travel of light and $\vec{u}_E$ is the velocity of emitter relative to receiver.
- Statistical & Quantum Mechanics
  - If the probability of $\vec{x}, \vec{p}$ is $P(\vec{x}, \vec{p})$, the average of quantity Q is

$$\langle Q \rangle = \int d^3\vec{x} \int d^3\vec{p}\, Q P(\vec{x}, \vec{p}) \text{ (modify as appropriate)}$$

  - Boltzmann factor $P \propto \exp[-E/kT]$ for a state of energy E
  - Maxwell distribution for velocity or speed in ideal gas

$$P(\vec{v})d^3\vec{v} = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} d^3\vec{v} \quad \text{or} \quad P(v)dv = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} dv$$

  - Planck's law, Wien's Law, Stefan-Boltzmann Law

$$\rho(\nu) = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} \ , \quad \lambda_{max} T = w \ , \quad \rho = \frac{4\sigma}{c} T^4$$

  - For calculations $h = 7 \times 10^{-34}$ Js, $k = 10^{-23}$ J/K, $w = 3 \times 10^{-3}$ mK, $\sigma = 6 \times 10^{-8}$ J/m^{2}sK^4
- Math
  - Gaussian integrals

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}} \ , \quad \int_{-\infty}^{\infty} dx\, x^2 e^{-ax^2} = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

  - Hyperbolic trig functions: $d\cosh\theta/d\theta = \sinh\theta$, $d\sinh\theta/d\theta = \cosh\theta$

$$\cosh^2\theta - \sinh^2\theta = 1 \ , \quad \cosh^2\theta + \sinh^2\theta = \cosh(2\theta) \ , \quad 2\sinh\theta\cosh\theta = \sinh(2\theta)$$