PHYS-3103 Homework 2 Due 13 Dec 2013

This homework is due **during the final project meeting** on the due date. You may instead email a PDF (prepared with LaTeX) to Dr. Frey in advance.

1. **Geometric T-Dualities on T^2**

   To describe a 2-dimensional torus T^2, consider 2 coordinates x, y with periodic identifications

   $$(x, y) \simeq (x + 2\pi\sqrt{\alpha'}N_x, y + 2\pi\sqrt{\alpha'}N_y) \ , \quad N_x, N_y \in \mathbb{Z}. \tag{1}$$

   The metric generally takes the form

   $$ds^2 = \frac{R^2}{\alpha'} \left|dx + \tau dy\right|^2 \ , \tag{2}$$

   where τ is a complex number with positive imaginary part (if the metric has no cross terms, τ is pure imaginary, and the torus is rectangular). τ is known as the complex structure modulus of the torus. R is the proper radius of the torus.

   (a) Consider a second torus with coordinate periodicity (1) and modulus $\tau' = \tau + 1$. Find new coordinates x', y' with the same periodicities (1) such that the metric written in the new coordinates has the original modulus τ. This shows that the two tori are identical.

   (b) Similarly, show that a torus with $\tau' = -1/\tau$ and radius $R' = R|\tau|$ is identical to the torus described in (2).

   (c) Closed strings on the original torus of metric (2) can have coordinate momentum and winding $p_{x,y} = n_{x,y}/\sqrt{\alpha'}$ and $w^{x,y} = m^{x,y}/\sqrt{\alpha'}$ for integers $n_{x,y}, w^{x,y}$. Argue that the correct string spectrum is given by

   $$M^2 = g^{ij}p_i p_j + g_{ij}w^i w^j + \frac{2}{\alpha'}\left(N^\perp + \tilde{N}^\perp - 2\right) \tag{3}$$

   for the metric g_{ij} given in (2) (just show that this formula gives the right result). How do the momentum and winding quantum numbers $n_{x,y}, m^{x,y}$ change under the transformations of parts (a,b)?

   (d) Finally, argue that the transformations of (a,b) can be composed to make the general transformation

   $$\tau \to \frac{a\tau + b}{c\tau + d} \tag{4}$$

   for any integers a, b, c, d with $ad - bc = 1$.

   There is actually a very similar structure for the $R \to \alpha'/R$ T-duality, where R^2 gives the imaginary part of a modulus ρ and the real part is given by the Kalb-Ramond field.

2. **Poincaré Patch for AdS** *from Zwiebach 23.6*

   We know that AdS_{n+1} spacetime can be described as a hyperboloid in a flat spacetime with signature $(n, 2)$ (that is, 2 time directions). Specifically, the embedding spacetime has metric

   $$ds^2 = -du^2 - dv^2 + dw^2 + dx^i dx^i \ , \quad i = 1, 2, \cdots n - 1 \ , \tag{5}$$

and the AdS spacetime is defined by the (Lorentz invariant) equation

$$-u^2 - v^2 + w^2 + x^i x^i = -R^2 \ , \tag{6}$$

where R is the radius of curvature of the AdS spacetime.

To parameterize this surface, introduce coordinates t, z, y^i with

$$v + w = \frac{R}{z} \ , \quad u = \frac{R}{z} t \ , \quad x^i = \frac{R}{z} y^i \ . \tag{7}$$

Since z appears in the denominator of the embedding coordinates, we require $0 < z < \infty$. Show that the metric of the AdS_{n+1} spacetime in these coordinates is

$$ds^2 = \frac{R^2}{z^2} \left(-dt^2 + dz^2 + dy^i dy^i \right) \ . \tag{8}$$

The conformal boundary is at $z \to 0$ and is clearly Minkowski; therefore, these coordinates are said to describe the *Poincaré patch* of AdS as opposed to *global* coordinates discussed in the text.