- Time-Dependence (4th Axiom) Note: Some aspects not in Griffiths much

In classical mechanics, we have Newton's laws / least action / Hamilton's equations to tell us how a system moves in phase space.

* In QM, a time-dependent state $|\Psi(t)\rangle$ evolves according to the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle \ ; \quad H = \frac{\vec{p}^2}{2m} + V(\vec{x}) \text{ usually.}$$

where H is the Hamiltonian operator.

- Stationary States
  - You can of course write the Schrödinger eqn in any basis you want:
    + $$i\hbar \frac{\partial}{\partial t}\Psi(\vec{x},t) = \langle \vec{x}|H|\Psi(t)\rangle = \frac{-\hbar^2}{2m}\nabla^2\Psi(\vec{x},t) + V(\vec{x})\Psi(\vec{x},t)$$
    + $$i\hbar \frac{\partial}{\partial t}\tilde{\Psi}(\vec{p},t) = \langle \vec{p}|H|\Psi(t)\rangle = \frac{\vec{p}^2}{2m}\tilde{\Psi}(\vec{p},t) + V[i\hbar \vec{\nabla}_{\vec{p}}]\tilde{\Psi}(\vec{p},t)$$
    + Or particularly the energy/Hamiltonian eigenbasis
    $$i\hbar \frac{\partial}{\partial t}\langle E_n|\Psi(t)\rangle = \langle E_n|H|\Psi(t)\rangle = E_n\langle E_n|\Psi(t)\rangle$$
    b/c H is Hermitian.
  - Consider this last decomposition with $\langle E_n|\Psi(t)\rangle = c_n(t)$
    + Then $$i\hbar \frac{\partial c_n}{\partial t} = E_n c_n(t) \Rightarrow c_n(t) = c_n^0 e^{-iE_nt/\hbar}$$
    + The state itself is
    $$|\Psi(t)\rangle = \sum_n c_n^0 e^{-iE_nt/\hbar}|E_n\rangle$$
    where
    $$H|E_n\rangle = E_n|E_n\rangle \Rightarrow \langle \vec{x}|H|E_n\rangle = \frac{-\hbar^2}{2m}\vec{\nabla}^2\psi_n(\vec{x}) + V(\vec{x})\psi_n(\vec{x}) = E_n\psi_n(\vec{x}).$$
    + We call this last eqn for each eigenstate the (time-independent) Schrödinger equation.
  - If the state $|\psi\rangle$ is an energy eigenstate, $|\Psi(t)\rangle = e^{-iE_nt/\hbar}|E_n\rangle$, probabilities and expectations are <u>time independent</u>
  $$|\langle\lambda|\Psi\rangle|^2 = |\langle\lambda|E_n\rangle|^2, \quad \langle \mathcal{O}\rangle = \langle E_n|\mathcal{O}|E_n\rangle$$
  So we call these <u>stationary states</u>

– <u>Ehrenfest's Theorem</u> : Expectation values obey classical physics

- For example, $\frac{d}{dt}\langle x\rangle = \langle p\rangle/m$, $\frac{d}{dt}\langle p\rangle = -\left\langle \frac{\partial V}{\partial x}\right\rangle$ in 1D
- In general, what's the time dependence of an expectation value?
  + Consider $\langle \mathcal{O}\rangle(t) = \langle \Psi(t)|\mathcal{O}(t)|\Psi(t)\rangle$ w/ explicit time dependence in $\mathcal{O}$.
  + Schrödinger's equation is

    $$\frac{d}{dt}|\Psi(t)\rangle = \frac{-i}{\hbar} H|\Psi(t)\rangle \ , \quad \frac{d}{dt}\langle\Psi(t)| = \frac{i}{\hbar}\langle\Psi(t)|H$$
  + The product rule gives

    $$\frac{d}{dt}\langle\mathcal{O}\rangle = \frac{i}{\hbar}\langle\Psi|H\mathcal{O}|\Psi\rangle + \langle\Psi|\frac{\partial\mathcal{O}}{\partial t}|\Psi\rangle - \frac{i}{\hbar}\langle\Psi|\mathcal{O}H|\Psi\rangle$$

    $$= \frac{i}{\hbar}\langle[H,\mathcal{O}]\rangle + \left\langle\frac{\partial\mathcal{O}}{\partial t}\right\rangle$$
  + The time dependence has both explicit (from the form of $\mathcal{O}$) and implicit (from quantum evolution) parts.
- Let's work out the example of a 1D system $H = \frac{p^2}{2m} + V(x)$
  + Neither x nor p have explicit time dependence.
  + $[H,x] = [p^2,x]/2m = -\frac{1}{m}[x,p]\,p = -\frac{i\hbar}{m}p$

    so $\quad \frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\left(-\frac{i\hbar}{m}\right)\langle p\rangle = \frac{\langle p\rangle}{m}$ as expected
  + Similarly, $[H,p] = -[p,V(x)] = i\hbar\frac{dV}{dx}$

    $$\Rightarrow \quad \frac{d\langle p\rangle}{dt} = -\left\langle\frac{dV}{dx}\right\rangle$$
- Why is this "classical behavior"? What does the commutator mean?
  + Hamiltonian classical mechanics defines the <u>Poisson bracket</u>

    $$\{f,g\} \equiv \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial p} \ , \quad \text{etc.}$$
  + You can see $\{x,p\}=1$ <u>Canonical quantization</u> says that a quantum theory comes from a classical one by

    $$\text{quantum} \rightarrow [A,B] = i\hbar\{A,B\} \leftarrow \text{classical}.$$

+ Of course, we should really derive classical mechanics as a limit of quantum mechanics, but this is a shorthand for the relationship

+ In Hamiltonian classical mechanics, observables satisfy

$$\frac{d\mathcal{O}}{dt} = \{\mathcal{O}, H\} + \frac{\partial \mathcal{O}}{\partial t} \xrightarrow[\text{quantization}]{\text{canonical}} \text{Ehrenfest's theorem}$$

(This is one form of Hamilton's equation)

— Different pictures of time dependence.

- We can define a time-evolution operator (aka propagator) as follows:

  + Write any state in the energy eigenbasis

$$|\Psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |E_n\rangle = \sum_n c_n e^{-iHt/\hbar} |E_n\rangle$$

$$= e^{-iHt/\hbar} \sum_n c_n |E_n\rangle = e^{-iHt/\hbar} |\Psi(0)\rangle$$

  + We can evolve any state from time $t=0$ to t by acting with $e^{-iHt/\hbar}$!

  + Note that $i\hbar \frac{d}{dt}\left(e^{-iHt/\hbar}|\Psi(0)\rangle\right) = H\left(e^{-iHt/\hbar}|\Psi(0)\rangle\right)$ agrees with the Schrödinger equation.

  + Leaving operators constant (except for explicit time dependence) and having states evolve is the Schrödinger picture.

- In this framework, expectation values are

$$\langle \mathcal{O}\rangle(t) = \langle \Psi(t)|\mathcal{O}|\Psi(t)\rangle = \langle \Psi(0)|e^{iHt/\hbar}\mathcal{O}e^{-iHt/\hbar}|\Psi(0)\rangle$$

This suggests that

  + We treat all states as time-independent $|\Psi\rangle = |\Psi(0)\rangle$

  + Define the evolution of operators as

$$\mathcal{O}(t) \equiv e^{iHt/\hbar}\mathcal{O}(0)e^{-iHt/\hbar}$$

  (leaving out explicit time dependence).

  + This is the Heisenberg picture. Can you work out Heisenberg's equation for $d\mathcal{O}/dt$?

- Energy-Time Uncertainty Principle.
  - This is more "squishy" than the Heisenberg uncertainty principle
  - Start as usual defining $\Delta E = \sigma_H$. What is Δt? Time $\neq$ operator.
    - For any observable $\mathcal{O}$, we know

$$\sigma_H \sigma_{\mathcal{O}} \geq \frac{1}{2} |\langle [H, \mathcal{O}] \rangle|$$

    - If $\mathcal{O}$ has no explicit time dependence, the rhs $= \frac{\hbar}{2} \left| \frac{d\langle \mathcal{O} \rangle}{dt} \right|$
    - Now suppose Δt is the time it takes an operator to change "a lot"

$$\sigma_{\mathcal{O}} \approx \left| \frac{d\langle \mathcal{O} \rangle}{dt} \right| \Delta t.$$

  - We get approximately $\Delta E \Delta t \geq \hbar/2$. Can use this in heuristic ways.

Quick review of formalism

States are normalized vectors $|\psi\rangle$: $\langle \psi | \psi \rangle = 1$
and overall phases make no physical difference $|\psi\rangle \sim e^{i\phi} |\psi\rangle$.

Important basis states $|\vec{x}\rangle$ = position eigenstates
$|\vec{p}\rangle$ = momentum eigenstates $|E_n\rangle$ = energy eigenstates

We know $\langle \vec{x} | \psi \rangle = \psi(\vec{x})$, $\langle x | p \rangle = e^{ipx/\hbar} / \sqrt{2\pi\hbar}$ (in 1D)
$\langle \vec{x} | p | \psi \rangle = -i\hbar \vec{\nabla} \psi(\vec{x})$.

Schrödinger's equation is $i\hbar \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle$
but can be written as an eigenvalue problem in "time-independent" form
$H|E_n\rangle = E_n |E_n\rangle$. Usually $H = \vec{p}^2/2m + V(\vec{x})$.

Probability of observing eigenvalue λ: $|\langle \lambda | \psi \rangle|^2$

Expectation value $\langle \mathcal{O} \rangle = \langle \psi | \mathcal{O} | \psi \rangle$.