- Quantum Statistics (Please see alternate derivation in reading)
  - Ensembles + Partition Functions in Stat. Mech:

    Want to understand how to find average properties of a system
    - Imagine that we average over an ensemble of identical systems. That's a lot like QM expectation value.
    - But we can consider different ways the system can interact with the environment
      - + A closed system (meaning E conserved, particle # N conserved) = <u>micro canonical</u> <u>ensemble</u>
      - + System exchanges energy with heat bath = <u>canonical</u> <u>ensemble</u>
      - + System exchanges energy + particles with heat bath = <u>grand</u> <u>canonical</u> <u>ensemble</u>
  - The probability of a system (from the ensemble) to be in state i is
    - + $P = e^{-E_i/kT}/Z$, $Z = \sum_i e^{-E_i/kT}$ = partition function in canonical ensemble. ↑ Boltzmann factor
    - + $P = e^{-(E_i-\mu N_i)/kT}/\mathcal{Z}$, $\mathcal{Z} = \sum_i e^{-(E_i-\mu N_i)/kT}$ = grand partition
    - + k = Boltzmann's constant, T = temperature, $\mu \equiv$ chemical potential.

      kT is related to average energy, μ to average particle number.

      Use units with $k=1$.
  - Calculate averages from partition functions
    - + Canonical $\langle E\rangle = \frac{1}{Z}\sum E_i e^{-E_i/T} = -\frac{1}{Z}\frac{dZ}{d(1/T)} = T^2\frac{d\ln Z}{dT}$
    - + Grand canonical

      $$\langle N\rangle = \frac{1}{\mathcal{Z}}\sum N_i e^{-(E_i-\mu N_i)/T} = \frac{T}{\mathcal{Z}}\frac{d\mathcal{Z}}{d\mu} = T\frac{d\ln\mathcal{Z}}{d\mu}$$
    - + Ensembles equivalent in large systems with small fluctuations $\langle \delta E^2\rangle, \langle \delta N^2\rangle, \ldots$

– Quantum Distribution functions

- Consider many indistinguishable particles that don't interact
  - + Each particle just fills a 1-particle state j w/energy ϵ_j
  - + The many-particle state is specified by # n_j of particles in each single particle state (since they're identical)

$$E = \sum_j n_j \epsilon_j \,, \quad N = \sum_j n_j \quad \text{for each many particle state.}$$

- The grand partition function becomes (of whole system)

$$\mathcal{Z} = \sum e^{-(E-\mu N)/T} = \sum_{\{n_j\}} \exp\left[-\sum_j n_j(\epsilon_j - \mu)/T\right]$$

$$= \left(\sum_{n_1} e^{-n_1(\epsilon_1-\mu)/T}\right)\left(\sum_{n_2} e^{-n_2(\epsilon_2-\mu)/T}\right)\cdots$$

- Bose-Einstein Distribution
  - + Any number of bosons can be in any given 1-particle state so n_j runs from 0 to ∞.
  - + Each sum is geometric series $\mathcal{Z} = \left(\frac{1}{1-e^{-(\epsilon_1-\mu)/T}}\right)\cdots$
  - + The Bose-Einstein distribution

$$f(\epsilon,\mu,T) \equiv \langle n_j \rangle = -T\frac{d\ln\mathcal{Z}}{d\epsilon_j} = \frac{e^{-(\epsilon-\mu)/T}}{1-e^{-(\epsilon-\mu)/T}} = \frac{1}{e^{(\epsilon-\mu)/T}-1}$$

  - + The Planck law can be written as $\epsilon f(\epsilon,\mu=0,T) \times$ (# states w/energy ϵ).

- Fermi-Dirac Distribution
  - + Each single fermion state has only 0 or 1 particles due to antisymmetry
  - + Then $\mathcal{Z} = \left(1+e^{-(\epsilon_1-\mu)/T}\right)\cdots$
  - + The Fermi-Dirac Distribution is $f(\epsilon,\mu,T) = \langle n_j \rangle = \frac{1}{e^{(\epsilon-\mu)/T}+1}$
  - + As $T \to 0$, exponential is 0 if $\epsilon < \mu$, ∞ if $\epsilon > \mu$ so

$$f(\epsilon;\mu,T=0) = \Theta(\mu-\epsilon)$$

That's just what happens in a free-electron (Fermi) gas if $\mu = E_F$.

- See HW for more