PHYS-4601 Homework 8 Due 7 Nov 2013

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Particle in a Box and Degeneracy** *Based on Griffiths 4.2*

   Consider a 3D square well with potential

   $$V(x,y,z) = \begin{cases} 0 & 0 < x,y,z < a \\ \infty & \text{otherwise} \end{cases} \quad . \tag{1}$$

   That is, the particle moves freely within a box with walls at $x = 0, a$, $y = 0, a$, and $z = 0, a$.

   (a) Find the wavefunctions and energies of the stationary states.

   (b) In 1D quantum mechanics, there is only one bound state for a given energy. In 3D, there can be more than one; we call stationary states with the same energy *degenerate*, and the number of states with a given energy is the *degeneracy*. Give the three lowest energy eigenvalues and their degeneracies.

   (c) Write the three lowest energy eigenvalues for a similar potential but with walls at $x = 0, 2a$, $y = 0, a$, and $z = 0, a$. What are the degeneracies?

2. **A Finite Spherical Box** *extended from Griffiths 4.9*

   Consider a particle of mass m in the spherically symmetric potential

   $$V(r) = \begin{cases} -V_0 & (r < a) \\ 0 & (r \geq a) \end{cases} \quad . \tag{2}$$

   In 1D quantum mechanics, any potential that goes to zero at infinity and is negative anywhere has at least one bound state. We will see that is not true in 3D.

   (a) Assume $\ell = 0$ and energy $E < 0$. Find a transcendental equation that determines E. What is the condition on V_0 that allows a bound state?

   (b) Use Maple to plot the ground state energies associated with the range $\pi^2\hbar^2/8ma^2 < V_0 < \pi^2\hbar^2/2ma^2$. Attach a printout of your code and the plot.

   (c) Use the numerical method of assignment 6 to verify the bound state energy for $V_0 = \pi^2\hbar^2/4ma^2$. Plot the stationary state wavefunction in this case. Attach a copy of your Maple code (only for your final energy and wavefunction, please).

3. **Electromagnetic Gauge Transformations** *Griffiths 4.61*

   Now that we're in 3D, we could imagine having an electromagnetic field. For a particle of charge q in potential Φ and vector potential $\vec{A}$, the Hamiltonian is

   $$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\Phi \ . \tag{3}$$

   The electric and magnetic field are

   $$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \ , \quad \vec{B} = \vec{\nabla} \times \vec{A} \ . \tag{4}$$

   For more details, see Griffiths problem 4.59.

(a) Show that the electromagnetic fields are invariant under *gauge transformations*. That is, show that the potentials

$$\Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \ , \quad \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \tag{5}$$

give the same $\vec{E}$ and $\vec{B}$ fields as Φ and $\vec{A}$, where Λ is any function of $\vec{x}$ and t.

(b) Since the Hamiltonian involves the potentials, it looks like we can't just make a gauge transformation in the quantum theory. However, assuming that a wavefunction $\Psi(\vec{x}, t)$ solves the time-dependent Schrödinger equation for potentials Φ and $\vec{A}$, show that

$$\Psi' = e^{iq\Lambda/\hbar}\Psi \tag{6}$$

solves the time-dependent Schrödinger equation for the potentials Φ' and $\vec{A}'$ given in (5).

This gauge invariance is a critical feature of the quantum mechanical theory of electromagnetism with profound consequences. We may explore aspects of it again in assignments.