PHYS-4601 Homework 6 Due 17 Oct 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Proofs About Stationary States**

   (a) *Rephrasing Griffiths 2.1(b)* Consider the spatial part of a stationary state $\psi(\vec{x})$ (that is, $\Psi(\vec{x},t) = \psi(\vec{x})e^{-iEt/\hbar}$). Show that $\psi(\vec{x})$ can be chosen real as follows. Argue that, for any ψ that solves the time-independent Schrödinger equation, so does ψ^*. Use that to show that the real and imaginary parts of ψ are also solutions with the same energy. Finally, argue that normalizability of ψ implies normalizability of its real and imaginary parts.

   (b) *Griffiths 2.2 rephrased* Suppose that the energy E of a stationary state in one dimension is less than the minimum value of the potential. Use the time-independent Schrödinger equation to show that the second derivative of the wavefunction always has the same sign as the wavefunction. Then use that fact to argue qualitatively that such a wavefunction cannot be normalized, proving by contradiction that E must be greater than the minimum value of the potential.

2. **A Step Up** *most of Griffiths 2.34 plus some*

   Consider the step potential

   $$V(x) = \begin{cases} 0 & x \leq 0 \\ V_0 & x > 0 \end{cases} \quad . \tag{1}$$

   Assume $V_0 > 0$. Consider scattering states with an incoming wave on the left and energy $E > V_0$. The particle moving in this potential has mass m.

   (a) Find the reflection coefficient. Express your answer as a function of E, V_0, m.

   (b) To understand reflection and transmission coefficients when the potential takes different values on either side of the barrier, we should think about conservation of probability. Specifically, if we write the wavefunction as $\psi_{inc} + \psi_{ref}$ to the left and ψ_{trans} to the right, the reflection and transmission coefficients should be given by $R = |j_{ref}/j_{inc}|$ and $T = |j_{trans}/j_{inc}|$. Using your results from the previous assignment, show that $T = (k'/k)|\psi_{trans}/\psi_{inc}|^2$ for this potential, where $k = \sqrt{2mE}/\hbar$ and $k' = \sqrt{2m(E-V_0)}/\hbar$.

   (c) Using your answer to part (b), find the transmission coefficient for $E > V_0$ and verify that $T = 1 - R$.

3. **Numerical Determination of Energy Eigenvalue** *some combination of Griffiths 2.54 and 2.51*

   Consider the potential

   $$V(x) = -\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax) \ . \tag{2}$$

   Find the ground state energy using the numerical "wag the dog" method with Maple, as follows:

   (a) Show that the Schrödinger equation for a bound state can be written in terms of dimensionless variables as

   $$\frac{d^2\psi}{d\xi^2}(\xi) + \left(2\operatorname{sech}^2(\xi) - \kappa^2\right)\psi(\xi) = 0 \ , \tag{3}$$

   where κ is a positive constant.

(b) The ground state should be an even function, so if we ignore normalization, we can choose initial conditions $\psi(0) = 1$, $\psi'(0) = 0$. Starting with $\kappa^2 = 0.99$, enter the Schrödinger equation and initial conditions into Maple and solve the ODE numerically over the range $\xi = 0 - 10$. Then plot the solution. You may find the following Maple code helpful (note that we rename variables for ease of typing):

```
with(plots):
schr := diff(u(x),x$2)+(2*sech(x)^2-0.99)*u(x) = 0
init1 := u(0) = 1
init2 := (D(u))(0) = 0
psi := dsolve({init1, init2, schr}, numeric, range = 0 .. 10)
psiplot := odeplot(psi)
display(psiplot)
```

Attach a printout of your code with results.

(c) What happens to the wavefunction as ξ gets large? By increasing your chosen value of κ^2 to 1.01, you should be able to get the "tail" of the wavefunction to flip over. Since the correct wavefunction should go to zero at large ξ, this means you have bracketed the correct eigenvalue for κ^2. Choose successively closer together values of κ^2 to find the eigenvalue down to three decimal places. What's the ground state energy?

(d) This potential is exactly solvable, and the (unnormalized) ground state wavefunction is $\psi(x) = \text{sech}(ax)$. Plot both your numerical wavefunction and the exact one together on one plot and attach a hardcopy of your code and plot.

(e) Now suppose we cut the potential in half to $V(x) = -(\hbar^2 a^2/2m)\text{sech}^2(ax)$. Using the same method, find the ground state energy. Numerically determined quantities should be good to three decimal places. Attach a plot of your unnormalized ground state wavefunction in the same dimensionless variables as above.