PHYS-4601 Homework 4 Due 3 Oct 2013

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Unitary Operators and Time Evolution**

   Since the normalization condition $\langle\Psi|\Psi\rangle = 1$ represents the fact that all probabilities sum to one, it is very important that this normalization be time-independent in any physical system. We'll investigate this statement.

   (a) Consider *unitary* operators U, which are operators that satisfy $U^\dagger = U^{-1}$. If we transform our Hilbert space so that $|\Psi'\rangle = U|\Psi\rangle$ for all states $|\Psi\rangle$ and unitary U, show that $\langle\Phi'|\Psi'\rangle = \langle\Phi|\Psi\rangle$.

   (b) Show that $U = \exp[iA]$ is unitary if the operator A is Hermitian (define the exponential by its power series). *Hint:* You may want to show that $(AB)^\dagger = B^\dagger A^\dagger$.

   (c) Consider two states $|\Psi(t)\rangle$ and $|\Phi(t)\rangle$ that both evolve via the time evolution operator. Use the results above to show that $\langle\Phi(t)|\Psi(t)\rangle$ (and therefore the norm of any state) is time-independent. This is why quantum mechanics must have *unitary time evolution.*

   (d) Of course, in the Heisenberg picture, we can alternately keep states time-independent and evolve operators via the unitary time evolution operator

   $$\mathcal{O}(t) = e^{iHt/\hbar}\,\mathcal{O}(0)\,e^{-iHt/\hbar} \tag{1}$$

   (we do not consider explicit time dependence). Show that this operator obeys the *Heisenberg equation*

   $$\frac{d\mathcal{O}}{dt}(t) = \frac{i}{\hbar}[H, \mathcal{O}(t)]\ . \tag{2}$$

2. **Time Dependence & Schrödinger in 1D**

   A particle of mass m is confined to the interval $0 < x < L$ in one dimension (with Dirichlet boundary conditions), and it has wavefunction

   $$\Psi(x,t) = Ae^{-i\alpha t}\sin\left(\frac{\pi x}{L}\right)\left[1 + e^{-i\beta t}\cos\left(\frac{\pi x}{L}\right)\right]\ . \tag{3}$$

   (a) Use the Schrödinger equation to determine the functional form of the potential energy $V(x)$ in the region $0 < x < L$.

   (b) Find β. Is it possible to determine the value of α without more information, and does it matter?

3. **Matrix Hamiltonian**

   Consider a 3D Hilbert space with Hamiltonian

   $$H \simeq E_0 \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \tag{4}$$

   in some basis. Work in this basis throughout the problem.

(a) Show that the time evolution operator is

$$e^{-iHt/\hbar} \simeq \cos(E_0 t/\hbar) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - i \sin(E_0 t/\hbar) \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \tag{5}$$

in this basis.

(b) Suppose the system starts out at time $t = 0$ in a state represented by $[1\ 0\ 0]^T$. At time $t = \hbar\pi/4E_0$, what is the probability that a measurement of the energy returns $+E_0$?