PHYS-4601 Homework 21 Due 3 Apr 2014

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Photons**

   As given in class, the Hamiltonian (ignoring vacuum energy) for the electromagnetic field is

   $$H = \sum_{\lambda,\vec{k}} \hbar\omega_{\vec{k}} a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k}) \ , \tag{1}$$

   where $\vec{k}$ runs over possible wavevectors and λ over the two polarization states (either linear or circular, for example). The creation and annihilation operators satisfy the commutator

   $$\left[a_\lambda(\vec{k}), a_{\lambda'}^\dagger(\vec{k}')\right] = \delta_{\lambda,\lambda'}\delta_{\vec{k},\vec{k}'} \tag{2}$$

   and other commutators vanish. These are very much like ladder operators for the harmonic oscillator. You may use anything you know about harmonic oscillators to help solve this problem without having to prove it again.

   (a) Consider a state with n photons of wavevector $\vec{k}$ and polarization λ (and no other photons), which we define as

   $$|n, \vec{k}, \lambda\rangle = \frac{1}{\sqrt{n!}}\left(a_\lambda^\dagger(\vec{k})\right)^n |0\rangle \tag{3}$$

   for vacuum state $|0\rangle$ which vanishes when acted on by any annihilation operator. Show that $|n, \vec{k}, \lambda\rangle$ is an energy eigenstate and find its energy.

   (b) It is sometimes stated that matter can create photons through spontaneous emission (emission of a photon into a vacuum) or stimulated emission (emission caused by the presence of an electromagnetic field). The probability amplitude for creation of a photon by matter is proportional to $\langle n+1, \vec{k}, \lambda| a_\lambda^\dagger(\vec{k}) |n, \vec{k}, \lambda\rangle$, where n is the initial number of photons with wavevector and polarization $\vec{k}, \lambda$. Find the ratio of the probability for stimulated emission (the initial photon number n is very large) to the probability for spontaneous emission (initial photon number $n = 0$).

2. **Quantum Gates**

   (a) Consider a qbit represented by a charged spin-1/2 particle (such as an electron or proton) so that the bit $|0\rangle$ is spin down and $|1\rangle$ is spin up. Write the Hadamard gate operator as a matrix in the usual basis. ***Extra Credit:*** Then show that this operator is the time evolution operator for the charged particle first exposed to the z component of a magnetic field for an appropriate length of time and then exposed to the y component for the right length of time.

   (b) Consider the same physical representation of a qbit and show that the quantum NOT gate can be implemented by the time evolution operator of the particle in the x component of a magnetic field for an appropriate length of time (up to an unphysical overall phase).

   (c) Finally, consider a 2 qbit system. Choose a basis for the 2 qbit Hilbert space and write the CNOT gate operator as a matrix in that basis and show that it is unitary.