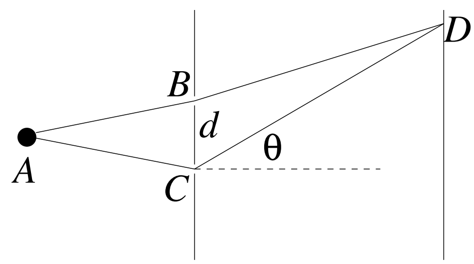
PHYS-4601 Homework 20 Due 27 Mar 2014

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **The Double-Slit Experiment and Aharonov-Bohm Effect**

   This problem will consider the classic double-slit experiment from the path integral point of view along with a version with a twist.

   (a) In the double-slit experiment illustrated below, two paths dominate the path integral for particles emitted at A and measured at D: from A to B to D with constant speed and from A to C to D at constant speed (the figure is not to scale).



   That means the amplitude for measurement of a particle at D is

$$\mathcal{M} = \int \mathcal{D}x\, e^{iS/\hbar} \approx e^{iS_{A\to B\to D}/\hbar} + e^{iS_{A\to C\to D}/\hbar} \ . \tag{1}$$

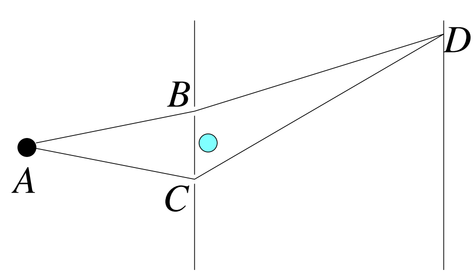
   Assume that the two slits at B and C are separated by a distance d which is very small compared to the distance between the two screens, so that segments BD and CD make the same angle θ with respect to the horizontal, and ignore the size of the slits. Show that the probability of observing the particle at D is proportional to $\cos^2(\pi d \sin\theta/\lambda)$, where $\lambda = 2\pi\hbar/p$ is the de Broglie wavelength. *Hint:* The length of each path ABD or ACD is much longer than the difference in path lengths, so you will want to do an expansion in the difference. Remember that both paths must last the same length of time, and use the total path length to determine the momentum.

   (b) Next, assume the particle has charge q but that the scalar potential is zero. Recall that the Lagrangian for a charged particle in a vector potential is $L = (m/2)\dot{\vec{x}}^2 + q\dot{\vec{x}}\cdot\vec{A}$ and show that the probability for measuring the particle at D becomes

$$P \propto \cos^2\left[\frac{\pi d}{\lambda}\sin\theta + \frac{q}{2}\oint \vec{A}\cdot d\vec{l}\right] \ , \tag{2}$$

   where the integral is from $A \to C \to D \to B \to A$.

   (c) Now consider a solenoid added to the experiment (the blue circle in the figure below) with a magnetic field pointed out of the paper. The particle cannot penetrate the solenoid.



Now use Stokes's Theorem (*aka* the curl theorem; see Griffiths's EM Theory textbook if necessary) to show that

$$P \propto \cos^2 \left[\frac{\pi d}{\lambda} \sin\theta + \frac{q}{2} \Phi_B \right] \ , \tag{3}$$

where Φ_B is the flux of the magnetic field through the solenoid. Amazingly enough, this shows that the magnetic field influences the interference pattern of the particle, even though the particle (or its wavefunction) cannot enter a region with a nonzero magnetic field! This is called the *Aharonov-Bohm effect*.

2. **Generating Functional**

Recall that the correlation function $\langle x(t_1)x(t_2)\rangle$ is defined in path integral form as

$$\langle x(t_1)x(t_2)\rangle = \int \mathcal{D}x x(t_1)x(t_2)e^{iS[x]/\hbar} \tag{4}$$

up to some (undefined) normalization constant. It is straightforward to extend this definition to to n operators:

$$\langle x(t_1)x(t_2)\cdots x(t_n)\rangle = \int \mathcal{D}x x(t_1)x(t_2)\cdots x(t_n)e^{iS[x]/\hbar} \ . \tag{5}$$

(a) Show that the general correlation function (or expectation value) (5) can be written as

$$\begin{aligned} \langle x(t_1)x(t_2)\cdots x(t_n)\rangle &= \left(-i\frac{\delta}{\delta j(t_1)}\right)\left(-i\frac{\delta}{\delta j(t_2)}\right)\cdots\left(-i\frac{\delta}{\delta j(t_n)}\right) Z[j]\Big|_{j=0} \\ \text{where } Z[j] &\equiv \int \mathcal{D}x \exp\left\{ iS[x]/\hbar + i\int dt j(t)x(t) \right\} \end{aligned} \tag{6}$$

using the functional derivative definition $\delta j(t')/\delta j(t) = \delta(t'-t)$ as in class. $Z[j]$ is called the *generating functional* because its derivatives generate the correlation functions.

(b) As a toy model of a path integral, imagine a world without time. In that world, "paths" $x(t)$ become just variables x, so path integrals are normal integrals. There is no kinetic energy, so the action for a "harmonic oscillator" is $S = -kx^2/2$. Expectation values are

$$\langle x^n \rangle \equiv \int_{-\infty}^{\infty} dx\, x^n \exp\left[-\frac{i}{2}kx^2 \right] \ , \tag{7}$$

and the generating functional is

$$Z[j] \equiv \int_{-\infty}^{\infty} dx\, \exp\left[-\frac{i}{2}kx^2 + ijx \right] \ . \tag{8}$$

Show that

$$Z[j] = Z[j=0]\exp\left[ij^2/2k \right] \ . \tag{9}$$

Then use the technique of part (a) to calculate $\langle x^4\rangle/Z[0]$ (the division gets rid of the arbitrary normalization).