PHYS-4601 Homework 2 Due 19 Sept 2013

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Boundary Conditions and Operators**

   Consider a particle in 1D confined to the line segment $0 < x < L$ (note that the Hamiltonian is not specified). Take the usual L^2 inner product. We will consider the three sets of boundary conditions on all wavefunctions: (i) Dirichlet $\psi(0) = \psi(L) = 0$ (ii) Neumann $d\psi/dx(0) = d\psi/dx(L) = 0$ (iii) periodic $\psi(x + L) = \psi(x)$. Answer each question below for each of those boundary conditions. The lesson of this question is that sometimes we have to be careful about boundary conditions.

   (a) Do functions with those boundary conditions and the L^2 inner product form a Hilbert space? Give only a brief argument, which may apply to all three boundary conditions.

   (b) Consider the momentum operator $p \simeq -i\hbar d\,/dx$ (which is another way of saying $\langle x|p|\psi\rangle = -i\hbar d(\langle x|\psi\rangle)/dx$). Is p an operator on the Hilbert space of functions with these boundary conditions?

   (c) Show that p^2 is a properly defined Hermitian operator in each case.

2. **Diagonalization** *Based on Griffiths A.26*

   Consider a three-dimensional Hilbert space with orthonormal basis $|e\rangle_i$, $i = 1, 2, 3$. The operator A takes the matrix representation

$$A = \sum_{i,j} |e_i\rangle\langle e_i|A|e_j\rangle\langle e_j| \simeq \begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix} . \tag{1}$$

   You should be able to check yourself that A is Hermitian.

   (a) Find the eigenvalues a_i and corresponding eigenstates $|a_i\rangle$ ($A|a_i\rangle = a_i|a_i\rangle$) written in terms of their components $\langle e_j|a_i\rangle$. Choose the eigenstates to form an orthonormal eigenbasis; that is, choose any ambiguities such that $\langle a_i|a_j\rangle = \delta_{ij}$.

   (b) Write the state $|\psi\rangle = |e_1\rangle - i|e_3\rangle$ in the A eigenbasis (as a superposition of the $|a_i\rangle$).

3. **Functions of Operators**

   (a) Suppose $|\lambda\rangle$ is an eigenfunction of $\mathcal{O}$, $\mathcal{O}|\lambda\rangle = \lambda|\lambda\rangle$. For any function $f(x)$ that can be written as a power series

$$f(x) = \sum_n f_n x^n \ , \tag{2}$$

   we can define

$$f(\mathcal{O}) = \sum_n f_n \mathcal{O}^n \ . \tag{3}$$

   Show that

$$f(\mathcal{O})|\lambda\rangle = f(\lambda)|\lambda\rangle \ . \tag{4}$$

   Does this result hold if the power series includes negative powers?

(b) What are the eigenstates $|\lambda\rangle$ and eigenvalues of the operator $T_a = \exp[-ipa/\hbar]$? Assume the states are functions on $-\infty < x < \infty$ and give the most general form for an eigenstate $|\lambda\rangle$. *Hint:* Do different momentum eigenstates have the same eigenvalue?

(c) Show that T_a translates a wavefunction by a distance a. That is, show that

$$T_a|\psi\rangle \simeq \psi(x-a) \text{ or equivalently } \langle x|T_a|\psi\rangle = \langle x-a|\psi\rangle \tag{5}$$

in the position basis. Based on this result, what is $T_a|x\rangle$? *Hint:* Think about the wavefunction's Taylor series around x.