PHYS-4601 Homework 19 Due 20 Mar 2014

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Uniform Gravitational Field** *parts of Griffiths 8.5 and 8.6*

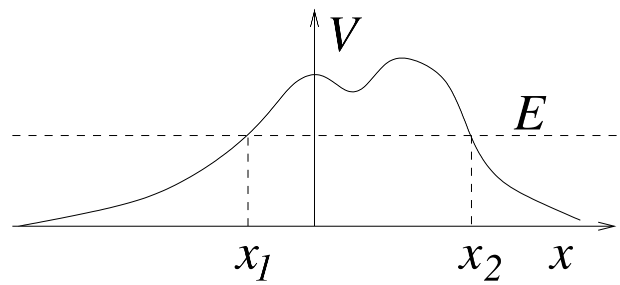
   Consider a ball of mass m that feels a uniform gravitational acceleration g in the $-x$ direction, as by the surface of the earth. Assume that the surface of the earth is at $x = 0$ and forms an infinite potential barrier.

   (a) First, write down what the potential energy is as a function of x.

   (b) Use the WKB approximation to find the allowed energies of the bouncing ball.

2. **Tunneling in the WKB Approximation**

   Consider a potential $V(x)$ in 1D as in the following figure (don't try to find the actual functional form of V) and a stationary scattering state with energy E as indicated by the dashed line.



   The classical turning points are x_1 and x_2. In the region to the left, where $E > V(x)$ and $x < x_1$, we can write the wavefunction in the WKB approximation as

$$\psi(x) = \frac{A}{\sqrt{p(x)}} \exp\left[-i\int_x^{x_1} dx'\, p/\hbar\right] + \frac{B}{\sqrt{p(x)}} \exp\left[i\int_x^{x_1} dx'\, p/\hbar\right] \ , \ p(x) = \sqrt{2m(E - V(x))} \ , \ x < x_1 \ , \tag{1}$$

   where A is the coefficient of the incident wave and B the coefficient of the reflected wave. Between the turning points,

$$\psi(x) = \frac{C}{\sqrt{\rho(x)}} \exp\left[-\int_{x_1}^{x} dx'\, \rho/\hbar\right] + \frac{D}{\sqrt{\rho(x)}} \exp\left[\int_{x_1}^{x} dx'\, \rho/\hbar\right] \ , \ \rho(x) = \sqrt{2m(V(x) - E)} \ , \ x_1 < x < x_2 \ . \tag{2}$$

   And finally, off to the right,

$$\psi(x) = \frac{F}{\sqrt{p(x)}} \exp\left[i\int_{x_2}^{x} dx'\, p/\hbar\right] \ , \quad x > x_2 \tag{3}$$

   is the transmitted wave.

   (a) Last semester, on assignment 6, you saw that the transmission coefficient is written as

$$T = \frac{|\psi_{trans}|^2 p_{trans}}{|\psi_{inc}|^2 p_{inc}} \ , \tag{4}$$

   where ψ_{trans} and ψ_{inc} are the transmitted and incident parts of the wavefunction and p_{trans}, p_{inc} are the momenta in the regions $x > x_2$ and $x < x_1$ respectively. Find T in terms of the coefficients A, B, C, D, F.

(b) Rewrite the connection formulas given in the class notes in terms of complex exponentials rather than sines and cosines. *Hint*: Note that the coefficients you use in the connection formulas are not necessarily those given in the wavefunction above.

(c) Use your new connection formulas as given in part (b) to write the coefficients C, D in terms of F and then A, B in terms of F. You should find that

$$A = \frac{1}{2}\left(2\theta + \frac{1}{2\theta}\right) F \text{ where } \theta = \exp\left[\int_{x_1}^{x_2} dx' \rho(x')/\hbar\right] . \tag{5}$$

Hint: As we did in class when studying bound states, you will find that the connection formulas give two different forms for the wavefunction in the region $x_1 < x < x_2$. You will need to relate those.

(d) Using your results from parts (a,c), find the transmission coefficient. In the limit that the barrier is high and wide, show that

$$T \approx \exp\left[-2\int_{x_1}^{x_2} dx' \rho(x')/\hbar\right] , \tag{6}$$

as we approximated in class.

3. **Charged Particle Lagrangian** *A common enough problem*

Recall from previous assignments that the Hamiltonian for a charged particle in a (given fixed) electromagnetic field is

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\Phi , \tag{7}$$

where q is the charge, Φ the scalar potential, and $\vec{A}$ the vector potential. Also remember that

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} , \quad \vec{B} = \vec{\nabla} \times \vec{A} . \tag{8}$$

(a) Show that Hamilton's equations give the Lorentz force law

$$m\ddot{\vec{x}} = q\left(\vec{E} + \dot{\vec{x}} \times \vec{B}\right) , \quad \dot{\vec{x}} = (\vec{p} - q\vec{A})/m . \tag{9}$$

Notice that the canonical momentum $\vec{p}$ is *not* the physical momentum! When you take time derivatives, be careful to include implicit as well as explicit time dependence. *Hint:* It will help to write these in components using index notation on derivatives. You will also like to know that

$$[\vec{a} \times (\vec{\nabla} \times b)]_i = \sum_j a_j \left(\frac{\partial b_j}{\partial x_i} - \frac{\partial b_i}{\partial x_j}\right) . \tag{10}$$

(b) Find the Lagrangian for the charged particle.

(c) Show that the canonical momentum as derived from the Lagrangian and Euler-Lagrange equations reproduce equation (9).