

PHYS-4601 Homework 16 Due 28 Feb 2014

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. Weak-Field Zeeman Effect

In the class notes, we stated that placing a hydrogen atom in a constant magnetic field $B_0 \hat{z}$ introduces a contribution to the hydrogen atom of $H_1 = (e/2m)B_0(L_z + 2S_z)$. If this contribution is larger than the energy level splitting due to fine structure, this gives the “strong-field” Zeeman effect that we discussed in class. In this problem, consider the opposite limit, in which H_1 is smaller than the fine structure splitting. In this case, we include the fine structure corrections in the “unperturbed” Hamiltonian H_0 and treat H_1 as the perturbation to that.

- (a) With fine structure included, the eigenstates of H_0 are identified by n , total angular momentum quantum number j , its z component m_j , and the total orbital angular momentum quantum number ℓ (as well as total spin $s = 1/2$); the z -components m_ℓ and m_s are not good quantum numbers. Write $H_1 = (e/2m)B_0(J_z + S_z)$ since $\vec{J} = \vec{L} + \vec{S}$ and show that the change in energy due to B_0 is

$$E_{n,j,m_j,\ell}^1 = -\frac{e\hbar}{2mc}B_0m_j \left[1 \pm \frac{1}{2\ell + 1} \right]. \quad (1)$$

To do this, you will need to know that the eigenstate of J^2 , J_z , and L^2 is written

$$\begin{aligned} |j = \ell \pm 1/2, m_j, \ell\rangle &= \sqrt{\frac{\ell \mp m_j + 1/2}{2\ell + 1}} |\ell, m_\ell = m_j + 1/2, m_s = -1/2\rangle \\ &\pm \sqrt{\frac{\ell \pm m_j + 1/2}{2\ell + 1}} |\ell, m_\ell = m_j - 1/2, m_s = 1/2\rangle \end{aligned} \quad (2)$$

in terms of the eigenstates of L^2 , L_z , and S_z . *Hint:* It may be useful to note that $[H_0, J_z] = [H_1, J_z] = 0$.

- (b) The quantity in square brackets in (1) is called the Landé g factor. Show that the g factor can also be written as

$$\left[1 + \frac{j(j+1) - \ell(\ell+1) + 3/4}{2j(j+1)} \right], \quad (3)$$

which is the form given in Griffiths. You can start with (3) and try $j = \ell \pm 1/2$ separately to get the form given in (1).

2. Hyperfine Structure *Griffiths 6.27*

Show that

$$\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \hat{r}_i \hat{r}_j = \frac{4\pi}{3} \delta_{ij}, \quad (4)$$

where \hat{r}_i is a component of the unit vector $\hat{r} = \sin\theta \cos\phi \hat{x} + \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z}$. Since the spherical harmonic $Y_0^0 = 1/\sqrt{4\pi}$ is constant, use this result to show that

$$\left\langle \frac{3(\vec{S}_p \cdot \hat{r})(\vec{S}_e \cdot \hat{r}) - \vec{S}_p \cdot \vec{S}_e}{r^3} \right\rangle = 0 \quad (5)$$

in $\ell = 0$ states of hydrogen (including the ground state). *Note:* You will not need to do any radial integrals in this problem!

3. Van der Waals Force *Griffiths 6.31 plus*

As you know, two atoms are attracted to each other because they are polarizable, and the induced dipoles attract. This is the Van der Waals force, which we will explore in a simple 1D model.

We consider two atoms, which we approximate as hydrogen (a positively charged nucleus and a single electron), moving in a line. This system has four coordinates: the center of mass coordinate (which we ignore as uninteresting), the relative separation R between the nuclei, and the separations x_1 and x_2 between the nucleus and electron of each atom. (See figure 6.14 in Griffiths). To make this model more tractable, we will pretend that R is frozen to a fixed value (which is reasonable since the heavier nuclear mass means that the nuclei move slowly compared to the electrons) and model each atom as a harmonic oscillator with Hamiltonian

$$H_0 = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_0^2x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0^2x_2^2. \quad (6)$$

That accounts for the Coulomb potential between the nucleus and electron *within* each atom; the Coulomb potential between the particles in separate atoms is given by the perturbation

$$H_1 = \frac{e^2}{4\pi\epsilon_0} \left(\frac{1}{R} - \frac{1}{R+x_2} - \frac{1}{R-x_1} + \frac{1}{R-x_1+x_2} \right). \quad (7)$$

As a side note, we can treat the electrons (and nuclei) as distinguishable particles because they are widely separated.

- (a) Assuming the inter-atomic spacing R is much larger than the intra-atomic spacings $|x_1|, |x_2|$, show that

$$H_1 \approx -\frac{e^2}{2\pi\epsilon_0} \frac{x_1x_2}{R^3} \quad (8)$$

to lowest order in $x_{1,2}/R$.

- (b) Find the *exact* ground state energy as follows: Show that the Hamiltonian $H = H_0 + H_1$ is that of two independent harmonic oscillators in the variables $x_{\pm} = (x_1 \pm x_2)/\sqrt{2}$ (and $p_{\pm} = (p_1 \pm p_2)/\sqrt{2}$) with frequencies

$$\omega_{\pm} = \sqrt{\omega_0^2 \mp (e^2/2\pi\epsilon_0 m R^3)}. \quad (9)$$

The ground state energy is therefore $E = \hbar(\omega_+ + \omega_-)/2$, while the ground state energy ignoring the interactions between atoms is $E^0 = \hbar\omega_0$. Show that the change in energy due to this van der Waals interaction is

$$\Delta E = E - E^0 \approx -\frac{\hbar}{8m^2\omega_0^3} \left(\frac{e^2}{2\pi\epsilon_0} \right)^2 \frac{1}{R^6} \quad (10)$$

to lowest order in the electric charge.

- (c) Now consider H_1 as a perturbation Hamiltonian and find the energy shift of the ground state at first and second order in perturbation theory. Confirm that you get the same result as part (b). Remember that the zeroth-order energy eigenstates are products of harmonic oscillator states, $|\psi_{n_1, n_2}^0\rangle = |n_1\rangle_1 |n_2\rangle_2$.

We've now seen that the Coulomb interaction between constituents of the atoms is responsible for an attractive $1/R^6$ potential. We can also look at the physical mechanism responsible for the attraction using perturbation theory.

- (d) Find the first-order correction to the ground state.
- (e) The Coulomb interaction can be thought of as the interaction between two aligned dipoles (as in the figure), but we should check that the dipoles are lined up. First, find the expectation value of a single dipole moment, $e\langle x_1 \rangle$. Then evaluate the dot product of the induced dipole moments, $e^2\langle x_1 x_2 \rangle$, in the perturbed ground state, to lowest order in e^2 (this is known as the *correlation* of the dipole moments). Explain your results.