PHYS-4601 Homework 14 Due 30 Jan 2014

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Bloch's Theorem**

   In class, we stated Bloch's theorem as saying that any stationary state wavefunction of a periodic potential $V(x+a) = V(x)$ can be written to satisfy the condition

   $$\psi(x+a) = e^{iKa}\psi(x) \ (K \text{ real}) \ . \tag{1}$$

   Alternately, Bloch's theorem can state that any stationary state wavefunction of the same potential can be written as

   $$\psi(x) = e^{iKx}u(x) \text{ where } u(x+a) = u(x) \ . \tag{2}$$

   (a) Show that these two formulations are equivalent (that is, show that if ψ satisfies (1) then it satisfies (2) and vice-versa).

   (b) If you write $u(x) = \sum_q c_q e^{iqx}$ as a Fourier series, what are the allowed values of q? Therefore, what are the allowed values of the momentum for a wavefunction with a given K?

2. **Specific Heat of the Free-Electron (Fermi) Gas**

   In this problem, you'll explore the thermal properties of the electrons in a simple metal. Recall that the total number of electrons and total energy of the gas (on average) are

   $$N = \sum_{states} f\left(\frac{\epsilon-\mu}{T}\right) \ , \quad U = \sum_{states} \epsilon f\left(\frac{\epsilon-\mu}{T}\right) \ , \quad f(x) = \frac{1}{e^x+1} \ . \tag{3}$$

   Here, T is the temperature, and μ is the chemical potential. For free electrons in volume V, the energy of each state is $\epsilon = \hbar^2k^2/2m$ in terms of the wavevector magnitude k, and there are $(Vk^2/\pi^2)dk$ states at that energy. It is a good approximation to replace the sum over states with integrals to define the number and energy densities

   $$n = \frac{1}{\pi^2}\int_0^\infty dk\, k^2 f(x) \ , \quad \rho = \frac{\hbar^2}{2m\pi^2}\int_0^\infty dk\, k^4 f(x) \ , \quad x = \frac{1}{T}\left(\frac{\hbar^2k^2}{2m} - \mu\right) \ . \tag{4}$$

   In this problem, you will find the specific heat of the free electron gas at low temperatures, which is defined as

   $$c_V = \frac{d\rho}{dT} \text{ with } \frac{dn}{dT} = 0 \ . \tag{5}$$

   Work only to lowest order in the temperature.

   (a) Argue that you can approximate

   $$\int_0^\infty dk\, k^2 g(x) = \frac{T}{2\mu}\left(\frac{2m\mu}{\hbar^2}\right)^{3/2}\int_{-\infty}^\infty dx\, g(x) \tag{6}$$

   for $x = (\epsilon-\mu)/T$ and any function $g(x)$ as $T \to 0$. Be careful with limits of integration.

   (b) Use conservation of particle density $dn/dT = 0$ to show that $d\mu/dT \to 0$ as $T \to 0$. Hint: Use your previous results and also the fact that df/dx is an even function of x.

(c) Show that

$$c_V = \frac{mT}{\hbar^3 \pi^2} \sqrt{2m\mu} \int_{-\infty}^{\infty} dx x^2 \frac{df}{dx} \ , \tag{7}$$

where the x integral is a pure number. To lowest order, you may set $d\mu/dT \to 0$ as follows from part (b).

(d) Finally, at $T = 0$, $\mu = E_F$, the Fermi energy, as discussed in class. Therefore argue that $c_V \propto T n^{1/3}$.