PHYS-4601 Homework 11 Due 9 Jan 2014

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Rotations** *parts of Griffiths 4.56*

   (a) Think back to our earlier problems on the translation operator. Argue that $\exp[i\varphi L_z/\hbar]$ is a rotation around the z axis by showing that

   $$e^{i\varphi L_z/\hbar} \cdot \psi(\phi) = \psi(\phi + \varphi) \tag{1}$$

   for any angular wavefunction $\psi(\phi)$ that can be written as a Taylor series around ϕ. *Hint*: This should be basically identical to what you did for the translation operator if you use the identification that $L_z = -i\hbar\partial/\partial\phi$.

   As a result, the angular momentum operators are called the *generators* of rotations. In general, $\hat{n} \cdot \vec{L}/\hbar$ generates rotations around the unit vector $\hat{n}$. Furthermore, the rotations of spinors are generated by the spin angular momentum operators.

   (b) What is the 2×2 matrix corresponding to a rotation of 2π around the z axis for spin 1/2? How does it compare to what you expected?

   (c) Construct the 2×2 matrix corresponding to a rotation of π around the x axis for spin 1/2. Show that it takes the S_z eigenstate $|+\rangle$ into $|-\rangle$.

2. **Hadron Spins** *Griffiths 4.35 plus*

   Quarks are elementary particles with spin 1/2, which we see in bound states called *hadrons*. Hadrons come in two varieties. In the following, assume that the quarks have zero orbital angular momentum.

   (a) *Mesons* are formed of a quark and antiquark (think of it as two quarks). What are the possible total spin quantum numbers of a meson?

   (b) *Baryons* are formed of three distinct quarks. What are the possible total spin quantum numbers? How many complete sets of states are there for each of those total spins?

3. **Spin Interactions**

   (a) Two spin 1/2 particles are fixed in position but have interacting spins. Their Hamiltonian is

   $$H = J\vec{S}^{(1)} \cdot \vec{S}^{(2)} \tag{2}$$

   for some constant J. Here $S^{(i)}$ is the spin operator of the ith particle. Find the energy eigenvalues of this system, their degeneracies, and the corresponding eigenstates. *Hint*: You will want to work in terms of the total spin quantum numbers.

   (b) The two spins have the same gyromagnetic ratio γ. In the presence of a magnetic field, the Hamiltonian becomes

   $$H = J\vec{S}^{(1)} \cdot \vec{S}^{(2)} - \gamma\vec{B} \cdot \left(\vec{S}^{(1)} + \vec{S}^{(2)}\right) \ . \tag{3}$$

   Now find the energy eigenvalues and their degeneracies. You may take $\vec{B}$ to lie along the z direction.

4. **Center of Mass Frame and Reduced Mass**

In class, we treat the hydrogen atom as if it is an electron moving around a stationary proton. Of course, that can't be, since it violates conservation of momentum. What happens, of course, is that the proton hardly moves in the center of mass rest frame. However, it turns out that we can always describe a system of two particles in terms of a single particle. In this problem, consider two particles of masses m_1 and m_2.

(a) In quantum mechanics, the kinetic energy is given by a Laplacian operator. Consider the 1D case for simplicity. Then the kinetic Hamiltonian is

$$H = -\frac{\hbar^2}{2m_1}\frac{d^2}{dx_1^2} - \frac{\hbar^2}{2m_2}\frac{d^2}{dx_2^2} \ , \tag{4}$$

where x_1 is the first particle's position and x_2 is the second particle's position. Show that this kinetic Hamiltonian can be written as

$$H = -\frac{\hbar^2}{2M}\frac{d^2}{dX^2} - \frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} \ , \tag{5}$$

where $M = m_1 + m_2$ is the total mass, $\mu = m_1 m_2/M$ is the reduced mass, $X = (m_1 x_1 + m_2 x_2)/M$ is the center of mass position, and $x = x_1 - x_2$ is the relative position.

The proof is essentially the same for the 3D Laplacian, and we then set the center of mass momentum to zero by choice of reference frame. Therefore, when we study the hydrogen atom, we are really using the reduced mass of the electron, which is nearly the electron mass because the proton is so much heavier than the electron.

(b) Imagine an atom made of an electron and a deuteron (nucleus made of a proton and neutron); this is a deuterium atom. Deuterium atoms are exactly like hydrogen atoms (in terms of energy eigenvalues) except the proton mass is replaced by the deuteron mass. Find the fractional difference in ground state energies $(E_D - E_H)/E_H$, where E_D is the deuterium ground state energy and E_H is the hydrogen ground state energy. First, give your answer in terms of the electron, proton, and deuteron masses to $\mathcal{O}(m_e/m_p, m_e/m_d)$ and then numerically using $m_e = 0.511$ MeV/c^2, $m_p = 938$ MeV/c^2, $m_d = 1876$ MeV/c^2 (give your answer to three significant digits).

5. **Hydrogen Radial Wavefunction & Laguerre Polynomials** *Related to Griffiths 4.10, 12*

Work for this problem should be done entirely in Maple. Show your work by attaching a printout of your code and results. You may explain your strategy on a separate sheet. In this problem, I will suggest Maple commands for you to use, but you will need to look up how to use them in Maple help.

(a) The radial wavefunction for hydrogen can be written as $u(\rho) = \rho^{\ell+1}e^{-\rho}v(\rho)$ in terms of a dimensionless radius ρ, where $v(\rho) = \sum_j c_j \rho^j$ satisfies the recursion relation

$$c_{j+1} = 2\left[\frac{j+\ell+1-n}{(j+1)(j+2\ell+2)}\right] c_j \tag{6}$$

and n is the radial quantum number. Use the `rsolve` command to find a solution for the coefficients c_j in terms of j, ℓ, n. Assume $c_0 = 1$ for ease of normalization.

(b) Try evaluating your solution (you can copy-paste the formula) for $n = 3, \ell = 0, j = 0$ (do *not* use the `subs` command). Most likely, Maple gives a solution that takes the form 0/0 for the values we need to evaluate. If you look at your solution, you should see a factor of the form $\Gamma(j+x)/\Gamma(x)$ for x some negative value (where the Γ function generalizes the factorial). Due to properties of the Γ function,

$$\frac{\Gamma(j+x)}{\Gamma(x)} = x(x+1)\cdots(x+j-1)\ j = 1,2,\cdots \quad (=1 \text{ for } j=0)\ . \tag{7}$$

Use `simplify[GAMMA]` to show this for $j = 4$. This product is known as the *Pochhammer symbol*, which is `pochhammer`(x, j) in Maple and show you get the right answer in Maple for $j = 4$ (you will have to `expand` and then `factor`). Write your solution, substituting in the Pochhammer symbol, as a function of j, n, ℓ, and check that you get the answer 1 for $n = 3, \ell = 0, j = 0$.

(c) Now use the `sum` command to make the polynomial $v(\rho)$ for the cases (i) $n = 3, \ell = 0$ (ii) $n = 4, \ell = 1$ (iii) $n = 4, \ell = 2$. Note that the series always terminates after $j = n-\ell-1$. Finally, in each case, divide your series by the associated Laguerre polynomial $L^{2\ell+1}_{n-\ell-1}(2\rho)$, which is given by the `LaguerreL` command. If your series is correct, you should find a constant.