

PHYS-4601 Homework 10 Due 21 Nov 2013

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. Pauli Spin Matrices *Related to Griffiths 4.26*

The Pauli spin matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (1)$$

are constructed to satisfy the commutator

$$[\sigma_i, \sigma_j] = 2i \sum_k \epsilon_{ijk} \sigma_k, \quad (2)$$

where ϵ_{ijk} is the Levi-Civita symbol (as defined in class or under equation (4.153) in Griffiths). That gives the spin-1/2 operators $\vec{S} = (\hbar/2)\vec{\sigma}$ the correct commutators. This problem will explore another property of the Pauli matrices.

(a) Show that the Pauli matrices also satisfy

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \mathbf{1}, \quad (3)$$

where δ_{ij} is the Kronecker delta symbol and $\mathbf{1}$ is the 2×2 identity matrix.

(b) The four matrices $\mathbf{1}$ and $\sigma_{x,y,z}$ are linearly independent, so any 2×2 matrix A can be written as

$$A = a_0 \mathbf{1} + a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z \quad (4)$$

for some complex numbers $a_{0,1,2,3}$. Show that

$$a_0 = \frac{1}{2} \text{Tr}(A), \quad a_i = \frac{1}{2} \text{Tr}(\sigma_i A) \quad (i = 1, 2, 3 = x, y, z). \quad (5)$$

Here, Tr is the *trace*, the sum of the diagonal elements of the matrix argument. *Hint*: The trace has the *cyclic property* $\text{Tr}(AB) = \text{Tr}(BA)$ for any two matrices A, B .

2. Other Spin Components *Almost Griffiths 4.29*

In this question, work in the basis where the S_z eigenstates are $|+\rangle = [1, 0]^T$ and $|-\rangle = [0, 1]^T$.

- Find the eigenvalues and eigenvectors of the operator $S_y = (\hbar/2)\sigma_y$ (see eqn (1) above).
- If you measure S_y on a single particle, what possible values could you measure? What are the probabilities of those values in the general state $|\psi\rangle = a|+\rangle + b|-\rangle$ (a, b are complex numbers satisfying the normalization condition $|a|^2 + |b|^2 = 1$).
- What is the expectation value of S_y in the state $|\psi\rangle$ of part (b)?

Hint: Griffiths does some similar calculations in the text for the operator S_x .