

## • The Invariant Interval

(14)

Beware: notation different than text! Particularly signs.

- Define the invariant interval  $\Delta s^2$  between 2 events by

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta \vec{x}^2 \quad \Delta \vec{x}^2 = \Delta x^2 + \Delta y^2 + \Delta z^2, \text{ etc.}$$

- This is like a Pythagorean distance but with a sign on the time part.
- What do we mean invariant?

It's the same when calculated in any <sup>inertial</sup> reference frame.

• Proof:

What we want to show is

$$-c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -c^2 \Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

for any frames  $S + S'$  (inertial)

+ Use standard configuration.

+ We know  $\Delta y = \Delta y'$ ,  $\Delta z = \Delta z'$ , so need

$$-c^2 \Delta t^2 + \Delta x^2 = -c^2 \Delta t'^2 + \Delta x'^2$$

+ Start in  $S'$  and use Lorentz transformations

$$\begin{aligned} -c^2 \Delta t'^2 + \Delta x'^2 &= -c^2 \gamma^2 \left( \Delta t - \frac{v \Delta x}{c^2} \right)^2 + \gamma^2 (\Delta x - v \Delta t)^2 \\ &= -c^2 \Delta t^2 \gamma^2 \left( 1 - \frac{v^2}{c^2} \right) + \Delta x^2 \gamma^2 \left( 1 - \frac{v^2}{c^2} \right) + 2\gamma^2 v \Delta t \Delta x \\ &\quad - 2\gamma^2 v \Delta t \Delta x \\ &= -c^2 \Delta t^2 + \Delta x^2 \quad \checkmark \end{aligned}$$

- We have 3 cases

- $\Delta s^2 > 0$ . Consider 2 events separated by  $\Delta \vec{x}, \Delta t$  with  $c \Delta t < |\Delta \vec{x}|$ .

Then you can go to a frame with  $\Delta t' = \gamma \left( \Delta t - \frac{v}{c^2} \Delta x \right) = 0$

How? Line up  $\Delta \vec{x}$  along  $x$ -axis. Choose  $v = c^2 \Delta t / \Delta x$ .

Then, in new frame,

$$\Delta s^2 = (\Delta x')^2$$

Call this the (squared) proper distance. This is space-like separation.

•  $\Delta s^2 < 0$ .

•  $\delta s^2 < 0$ . This is time-like separation.

We can choose a frame where  $\delta \vec{x}' = 0$ . So  $\delta s^2 = -c^2 \delta t'^2$ .

We define  $\delta \tau^2 = -\delta s^2/c^2 = +\delta t'^2 + \delta \vec{x}'^2/c^2 = (\delta t')^2$

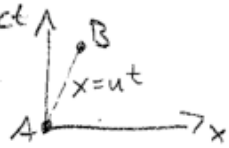
$\delta \tau$  is the proper time between the two events.

•  $\delta s^2 = 0$ . Imagine a light ray. Since light has  $|\delta \vec{x}| = c \delta t$ ,  
 $\delta s^2 = -c^2 \delta t^2 + \delta \vec{x}^2 = -c^2 \delta t^2 + c^2 \delta t^2 = 0$ . That means light can travel between the two events. They are light-like separated.

- Examples

• Suppose A and B are timelike separated events. Then  $|\delta \vec{x} / \delta t| < c$   
Some object moving at speed  $u < c$  can go from A to B. Then

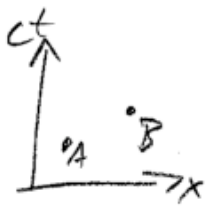
$$\delta s^2 = -c^2 \delta t^2 + \delta \vec{x}^2 = -c^2 \delta t^2 + u^2 \delta t^2$$



The proper time of the object's clock is  $\delta \tau = \sqrt{-\delta s^2/c^2} = \delta t \sqrt{1 - u^2/c^2}$   
Familiar?

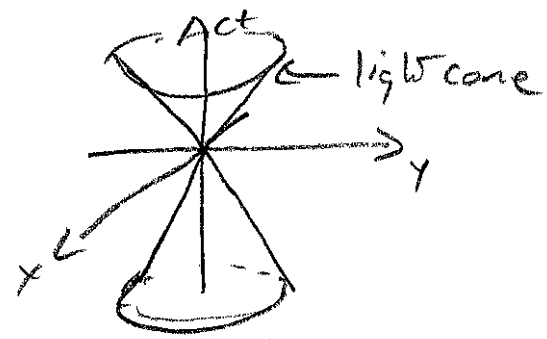
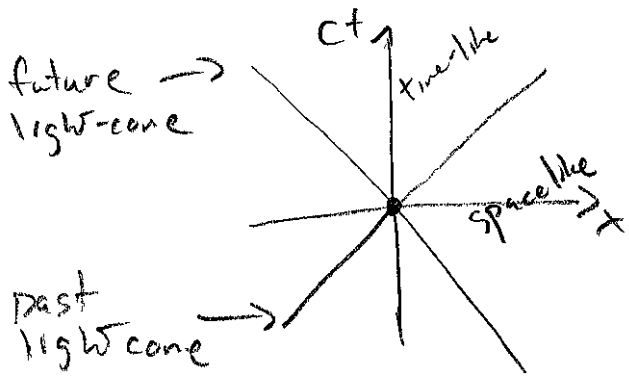
• Space time Diagrams

- For a while now, we've drawn diagrams as to the right, indicating events as points. But we can do more!



- First, if we take the origin as one event, we can divide the diagram into time-like, space-like, and light-like regions

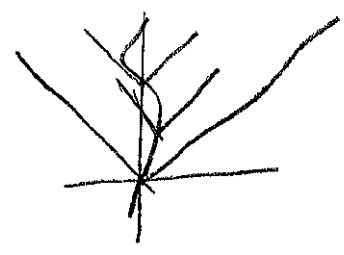
Note: light travels at  $45^\circ$



- The light-cone is all the light-like separated events.
- From an event, you can send a signal to anything on or in your future light cone
- And you can receive a signal from anywhere on or in your past light cone
- No matter your frame, the light-cone is at  $45^\circ$ !

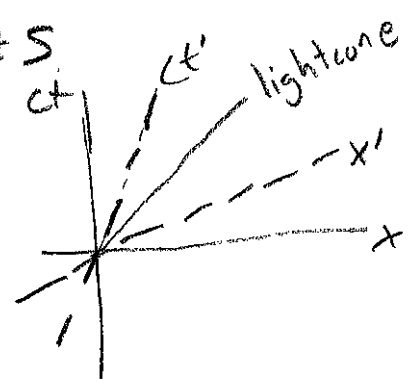
- D. We can draw the trajectories of objects/particles through time. These are called world lines.

World lines can never bend out of the light cone at any point  
Useful for illustration

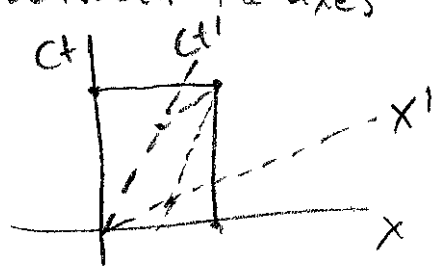


- Different frames in one diagram:

- Take  $S'$  in standard configuration w.r.t  $S$
- What's the  $t'$  axis? That's just where  $x'=0$ , or  $ct = vx (c/v)$
- How about  $x'$  axis? That's where  $t'=0$ , or  $ct = (v/c)x$



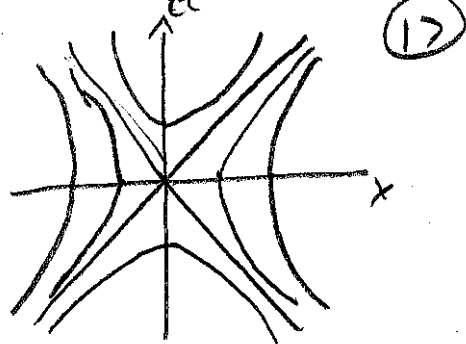
- The light cone is still halfway between the axes
- To get coordinates, draw lines parallel to axes



- Other invariant curves:

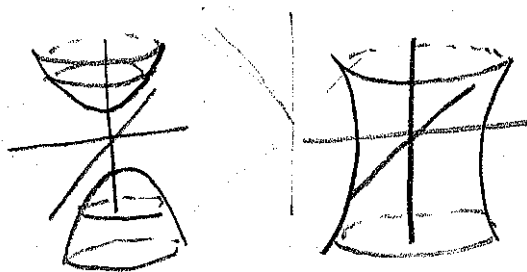
- Just like  $Ss^2$  is invariant,  
 $c^2 t^2 - x^2$  is invt

(From the same Lorentz transformations)



- Curves where this = 0 are lightcones
- Otherwise =  $\pm(\neq)$  are hyperboloids (hyperbolas in 2D)

Note that these rotate into  $y, z$   
as well. like lightcones



- By a Lorentz transformation,

you can move any point on a hyperboloid to any other point on the same hyperboloid

- + Shows that spacelike separated events have no notion of "past", "future," or "simultaneity". You can change time order
- + But timelike separated events can't change order!

There is a notion of past and future, but only inside the light-cone.

- This shows us the causal structure of spacetime:

which events can cause something to happen at other events.

+ Note: That's why you can't travel faster than  $c$ .

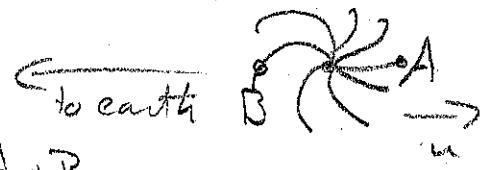
If you could get outside the light cone, you could change frames and "time-travel" to the past.

Then you could kill your grandparents before your parents are born!

# Examples of Spacetime Diagrams

Back to the Moving Galaxy and 2 supernovae

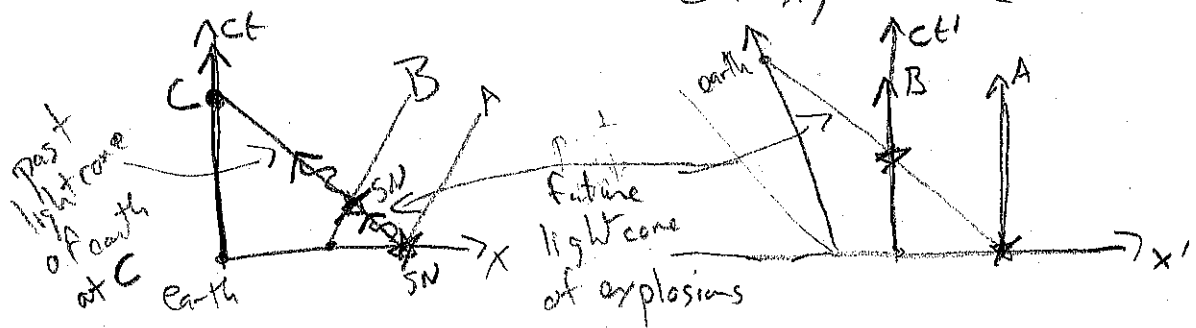
Recall the problem: Galaxy moves away from us at constant speed.



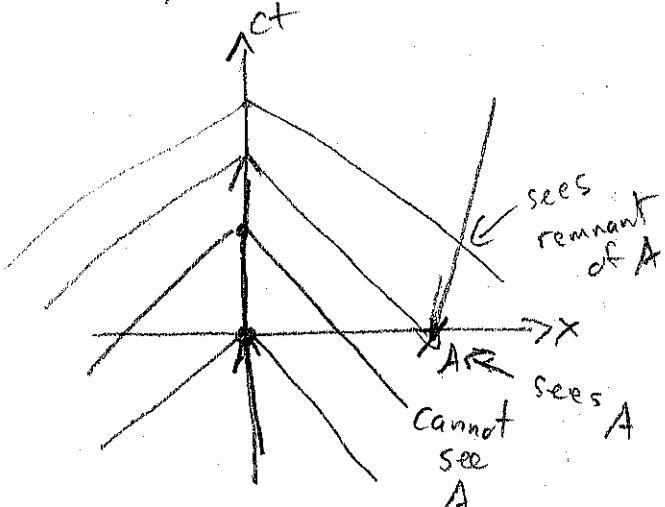
Explosions occur at spacetime points A+B  
Light from them reaches earth at the same time

Earth's frame

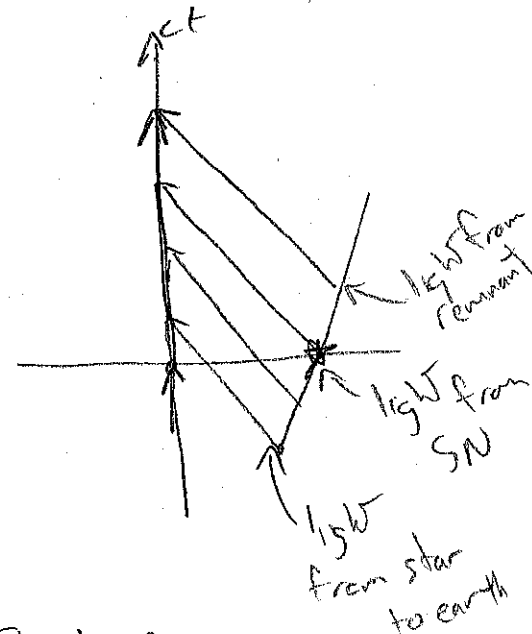
Galaxy's frame



One more point. Earth's worldline + explosions are spacelike separated at some points, then lightlike + timelike separated



Earth's past lightcones



Star's future lightcones

Just looking at the star/SN from A.

Hopefully this gives you a picture of the problem.