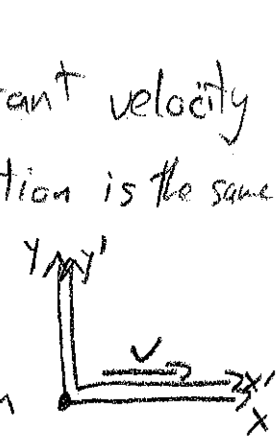
Lorentz Transformations + Spacetime

- Transformation of coordinates by shift of constant velocity

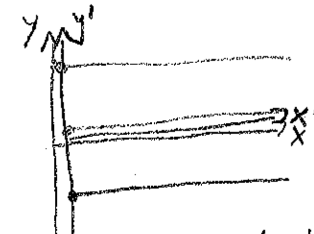
Of course, transformation by translation or rotation is the same as before

— We will use the "standard configuration" with velocity along x. Others are related by rotation



— First look at orthogonal directions y, z and transformations of them

- You can do a thought experiment by putting paint brushes at tick marks of y and y' so they mark out lines in S and S' frames



- Lines must be parallel to motion or you can tell position (violate translation)
- and must coincide or you can tell moving left from moving right (violates rotation symmetry since related by 180° rotation)

$$\Longrightarrow \quad y = y', \quad z = z'$$

— Now look at transformation of (x, t) to (x', t')

- An object at rest in S' cannot accelerate in S (+ vice versa)

In fact, it must move at constant velocity v along x.

so $\qquad x - vt = \text{const} \propto x'$, $\quad x' + vt' \propto x$

These are linear relationships. This is necessary. Write them as

$$x' = \gamma(v)(x - vt) \qquad x = \gamma(-v)(x' + vt')$$

- If motion to left and motion to right are not distinguished, we need $\gamma(v) = \gamma(-v)$ ie γ is an even function of v.
- Now we use the postulate of an invariant speed c.

(Note: set $c \to \infty$ for Galilean transformation)

Imagine a signal of speed c (flash of light) set off at $(x, t) = (x', t') = 0$. Then, the signal moving along $+x$ has

$$x = ct \quad . \quad x' = ct'$$

- Algebra requires $ct' = \gamma(v)(ct - vt)$ $ct = \gamma(v)(ct' + vt')$ (10)

$$\Rightarrow \quad ct' = \gamma(c-v)\left[\frac{\gamma}{c}(c+v)t'\right] \Rightarrow c^2 = \gamma^2(c^2 - v^2)$$

or $$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$ Called relativistic γ factor

- So

$$x' = \gamma(x - vt), \quad x = \gamma(x' + vt'), \quad \gamma = 1/\sqrt{1 - v^2/c^2}$$

More algebra gives

$$t' = \frac{x/\gamma - x'}{v} = \frac{1}{v}\left[x(1/\gamma - \gamma) + \gamma vt\right] = \gamma\left(t - \frac{vx}{c^2}\right)$$

Show $\frac{1}{v}\left(\frac{1}{\gamma} - \gamma\right) = \frac{1}{v}\frac{1}{\sqrt{1 - v^2/c^2}}\left(1 - v^2/c^2 - 1\right) = -\gamma v/c^2$.

- All told

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$
$$t' = \gamma(t - vx/c^2) \qquad t = \gamma(t' + vx'/c^2)$$
$$y = y', \quad z = z'$$

Lorentz boost transformations

Book gives slightly different derivation

- If $c \to \infty$, you get Galilean boost transformations

— When is it important to use these? For $x \lesssim vt$, the corrections are order v^2/c^2

Suppose we change into the rest frame of a moving object $u = v$

- Speed of sound in air $u \simeq 300\,m/s$, $(u/c)^2 \simeq 10^{-12}$
- Speed of earth in orbit $u \simeq 30\,km/s$, $(u/c)^2 \simeq 10^{-8}$
- Speed of electron in ground state of Hydrogen, $(u/c)^2 \simeq 5 \times 10^{-5}$

"nonrelativistic"

- Electron accelerated by supernova
- Protons at Large Hadron Collider $(u/c)^2 \lesssim 1$

"Ultrarelativistic"

- Consequences for Structure of Space + Time

Time transforms as well as space! Ultimately has to do with clock synchronization. Speak of spacetime

- Also works for small intervals $\delta x' = \gamma(\delta x - v\delta t)$

etc.

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— Let's revisit time dilation. Consider a clock located at $x'=0$ for all t'. What frame gives proper time? (S')

- How much time δt elapses for a change $\delta t'$ of proper time?

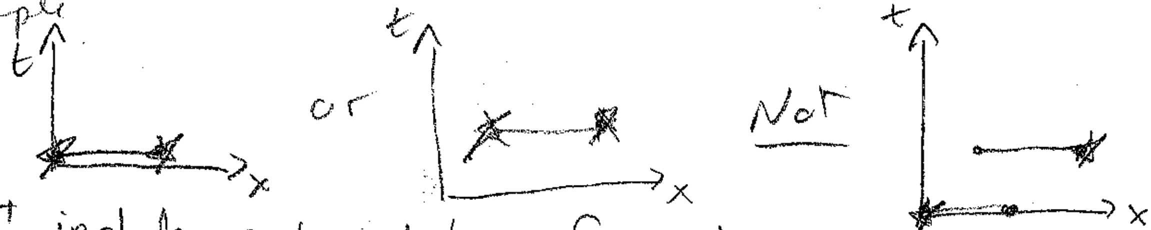
$$\delta t = \gamma(\delta t' + v\delta x'/c^2) = \gamma \delta t' > \delta t'$$

- This again shows that the "moving clock" in S' runs "slow"
- Although clocks of S are synchronized, you are measuring 1 clock in S' vs multiple clocks in S (because $\delta x = v\delta t$)
- Called time dilation because time in other frames increases compared to proper

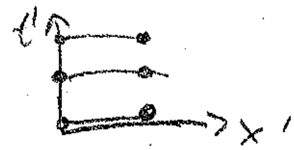
— Lengths also change. Let's look.

- First of all, what is length? We mean the distance between two sides of an object at a single time w.r.t. our reference frame

For example



- We don't include extra distance from motion
- Let S' be the rest frame of an object lying along x' axis. Moves at velocity v compared to S in std. configuration



Length is $l_0 = \delta x'$ = change in x' for $\delta t' = 0$
proper length

- The length in S is measured with $\delta t = 0$.

But we know $\delta x' = \gamma(\delta x - v\delta t) = \gamma \delta x$ for length δx.

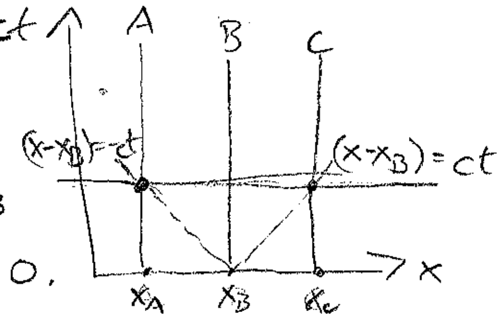
That means the length is smaller in S : $\delta x = \delta x'/\gamma = l_0/\gamma$

- This is length contraction

- This is not just how something looks. It's a real effect!

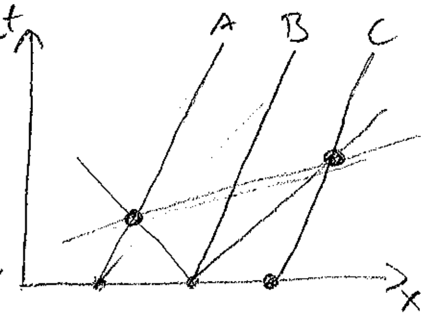
How something looks (to your eyes or a camera) is light leaving the different parts of the object at different times + arriving at us at same time.

- Simultaneity and Clock Synchronization
  - Apparently a boost by $v > c$ gives imaginary γ factor. That makes no sense. So that means no particle motion can be faster than c. Must synchronize clocks with light signals
  - Consider the frame S. We put detectors at ~~r~~est at x_A, x_B, x_C. Their These are evenly spaced $x_B - x_A = x_C - x_B$. B detector emits flash of light at $t=0$.



Light travels outward, hits A and C. These events are simultaneous and define an "x-axis" at $t = \frac{x_C - x_B}{c} = \frac{x_B - x_A}{c}$

  - But what if detectors are at rest in S' moving to right at speed v?



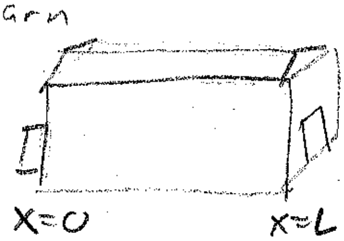
The detector positions x increase linearly in t. but are still equidistant in S' $(x'_B - x'_A = x'_C - x'_B$.

Can you show by Lorentz transformations? Just like length contraction

But now light hits A and B at different times t.

Nonetheless, those events must be at the same time in S'

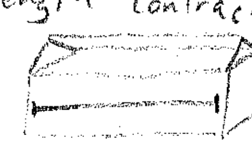
  - Synchronization and simultaneity are different in different frames!
- Another example using Lorentz transformations: Pole in Barn
  - You have a barn of length L with doors on either end. Front door starts out open, back door shut



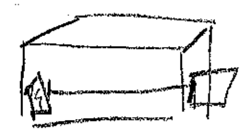
  - A pole of proper length $\frac{5}{4}L$ moves at speed $\frac{3}{5}c$ to the right. You (in the barn) think it is length contracted to

$$\left(\frac{5}{4}L\right)\sqrt{1-(3/5)^2} = L$$

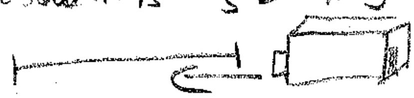
so it just fits.



- When the back end of the pole reaches the front door, Some one in barn closes front door.
  (At same time) When front part of pole reaches back door, someone else in barn opens back door.
  That way, pole does not crash (maybe it's a Delorean instead)



- But what do I see sitting on the pole (or in the car)?
  My pole is $\frac{5}{4}L$ long, but the barn is $\frac{4}{5}L$ long in my frame.
  How does it fit?



- Work out Lorentz transformations, + Set $t=0$ to front of pole at back barn door. That's at $x=L, t=0$.
  In pole frame, $x' = \gamma(x - vt) = \frac{5}{4}L$. $t' = \gamma\left(t - \frac{vx}{c^2}\right) = -\frac{3}{4}\frac{L}{c}$
  + When does back of pole reach front door? In barn frame, $x = t = 0$.
    In pole's frame $x' = t' = 0$ (overlapping origins)
  + The two events (door opening + closing) are <u>not simultaneous</u>
    The back door opens before the front door closes!
  + In fact, in pole's frame, $\delta x' = \frac{5}{4}L - \frac{4}{5}L = \frac{9}{20}L$. But that is $v\delta t'$.
    So the back of the pole just makes it in before the front door closes