PHYS-3301 Homework 7 Due 30 Oct 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Some Short Practice Calculations**

   Calculate the following quantities. You should get a number for each answer.

   (a) $\eta_{\mu\nu}\eta^{\mu\nu}$

   (b) $\eta^{\mu\nu}\eta^{\lambda\rho}\epsilon_{\mu\nu\lambda\rho}$

   (c) $\epsilon_{\mu\nu\lambda\rho}\epsilon^{\mu\nu\lambda\rho}$

   Based on a problem by Sean Carroll In the next two calculations, define the tensor and vector

$$\left[\quad X^{\mu\nu} \quad \right] = \begin{bmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{bmatrix} , \quad V^{\mu} = (-1, 2, 0, -2) \ . \tag{1}$$

   Then calculate the following:

   (d) $X^{\mu}{}_{\mu}$

   (e) $X^{\mu\nu}V_{\mu}V_{\nu}$

2. **Derivatives Have Lowered Indices**

   As discussed in the class notes, 4-vectors with raised or lowered indices have the following Lorentz transformations:

$$a^{\mu'} = \Lambda^{\mu'}{}_{\nu} a^{\nu} \quad \text{and} \quad a_{\mu'} = \bar{\Lambda}_{\mu'}{}^{\nu} a_{\nu} \ , \tag{2}$$

   where Λ is the usual Lorentz transformation matrix and $\bar{\Lambda}^T = \Lambda^{-1}$ as a matrix.

   (a) Show that the matrix relationship between $\bar{\Lambda}$ and Λ may be written as $\bar{\Lambda}_{\mu'}{}^{\rho}\Lambda^{\nu'}{}_{\rho} = \delta^{\nu'}_{\mu'}$ and $\bar{\Lambda}_{\rho'}{}^{\mu}\Lambda^{\rho'}{}_{\nu} = \delta^{\mu}_{\nu}$, where $\delta^{\nu'}_{\mu'}$ and δ^{μ}_{ν} are Kronecker delta symbols.

   (b) Using the fact that the spacetime position x^{μ} is a 4-vector, find the partial derivatives $\partial x^{\mu}/\partial x^{\nu'}$ and $\partial x^{\mu'}/\partial x^{\nu}$ in terms of $\Lambda^{\mu'}{}_{\nu}$ and $\bar{\Lambda}_{\mu'}{}^{\nu}$. *Hint:* For two positions as measured in the same frame, $\partial x^{\mu}/\partial x^{\nu} = \delta^{\mu}_{\nu}$ (think about why).

   (c) If f is a Lorentz invariant function (meaning its value at a fixed spacetime point is the same in any frame — like the temperature), use the chain rule to show that

$$\frac{\partial f}{\partial x^{\mu'}} = \bar{\Lambda}_{\mu'}{}^{\nu} \frac{\partial f}{\partial x^{\nu}} \ . \tag{3}$$

   In other words, you are showing that a partial derivative has the same transformation as a 4-vector with a lowered index. As a result, people will usually write $\partial_{\mu} f \equiv \partial f/\partial x^{\mu}$.

3. **Energy-Momentum Tensor**

   The energy-momentum tensor $T^{\mu\nu}$ is a relativistic tensor that describes the energy density, pressure, and stress forces of a fluid. In the rest frame S of a perfect fluid, $T^{00} = \rho$ (the energy density) and $T^{11} = T^{22} = T^{33} = p$ (the pressure), with all other components equal to zero. In other words,

$$\begin{bmatrix} & & & \\ & T^{\mu\nu} & & \\ & & & \end{bmatrix} = \begin{bmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix} \tag{4}$$

(blank elements are zero). Now consider a frame S' moving at speed v with respect to S in standard configuration. Under what condition on ρ and p is the stress tensor the same in S' as in S ($[T^{\mu'\nu'}] = [T^{\mu\nu}]$ as a matrix)? *Hint:* Remember that the Lorentz transformation of a tensor transforms each index independently:

$$T^{\mu'\nu'} = \Lambda^{\mu'}{}_{\alpha}\Lambda^{\nu'}{}_{\beta}T^{\alpha\beta} \ . \tag{5}$$