

PHYS-3301 Homework 6 Due 23 Oct 2013

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. Boosts and Rotations

In matrix form, we can define the boost Λ_{tx} along x and the rotation Λ_{xy} in the xy plane (around the z axis) as follows:

$$\Lambda_{tx}(\phi) = \begin{bmatrix} \cosh \phi & -\sinh \phi & & \\ -\sinh \phi & \cosh \phi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad \Lambda_{xy}(\theta) = \begin{bmatrix} 1 & & & \\ & \cos \theta & \sin \theta & \\ & -\sin \theta & \cos \theta & \\ & & & 1 \end{bmatrix}. \quad (1)$$

Empty elements in the matrices above are zero.

(a) In matrix form, the metric $\eta_{\mu\nu}$ is

$$\eta = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \quad (2)$$

Show that both rotation and boost in (1) satisfy the condition $\eta_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta}$, which is $\eta = \Lambda^T \eta \Lambda$ in matrix notation.

- (b) Consider two successive boosts along x , $\Lambda_{tx}(\phi_1)$ and $\Lambda_{tx}(\phi_2)$. Show that these multiply to give a third boost $\Lambda_{tx}(\phi_3)$ and find ϕ_3 . Using the relationship $v/c = \tanh \phi$ between velocity and rapidity ϕ , reproduce the velocity addition rule. *Hint:* You will need the angle-addition rules for hyperbolic trig functions.
- (c) First, write down the Lorentz transformation matrix $\Lambda_{ty}(\phi)$ corresponding to a boost along the y direction by permuting axes. Then show that you can get a boost along y by rotating axes, boosting along x , then rotating back by proving that $\Lambda_{ty}(\phi) = \Lambda_{xy}(-\pi/2)\Lambda_{tx}(\phi)\Lambda_{xy}(\pi/2)$.

2. 4-Vectors and Changing Frames

For parts (a,b) define a 4-vector U^μ to have components $U^0 = \cosh \phi$, $U^1 = \sinh \phi$, $U^2 = U^3 = 0$ in some reference frame S .

- (a) Now boost to reference frame S' which has velocity $v = c \tanh \theta$ along the x direction relative to S . What are the components of U^μ in S' ? Write your answer in terms of the "angle" $\phi - \theta$.
- (b) If $\phi - \theta$ is itself a rapidity, write the associated velocity in terms of $\tanh \theta$ and $\tanh \phi$.

Now we consider some arbitrary 4-vector a^μ .

- (c) If a^μ is timelike, show that there exists an inertial frame S where the only nonzero component is a^0 (that is, the spatial part \vec{a} is zero).
- (d) Suppose the components of a^μ in frame S are $a^0 = k$, $a^1 = k$, and $a^2 = a^3 = 0$. If frame S' is related to S by a boost of velocity v along the x direction, show that the components in frame S' are $a^{0'} = k'$, $a^{1'} = k'$, and $a^{2'} = a^{3'} = 0$, where $k' = \sqrt{(1 - v/c)/(1 + v/c)} k$.

3. Some Scalar Products

inspired by a problem in Hartle In some frame, the components of two 4-vectors are

$$a^\mu = (2, 0, 0, 1) \text{ and } b^\mu = (5, 4, 3, 0) . \quad (3)$$

- (a) Find a^2 , b^2 , and $a \cdot b$.
- (b) Does there exist another inertial frame in which the components of a^μ are $(1, 0, 0, 1)$?
What about b^μ ?

Now consider two 4-vectors a^μ and b^μ such that $a \cdot b = 0$. *based on Barton 7.5*

- (c) If a^μ and b^μ are both lightlike, prove that they are parallel ($a^\mu \propto b^\mu$).
- (d) Suppose $a^2 = -b^2$. Is the 4-vector $a^\mu + b^\mu$ spacelike, timelike, or lightlike?