PHYS-4601 Homework 8 Due 1 Nov 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Particle in a Box and Degeneracy** *Based on Griffiths 4.2*

   Consider a 3D square well with potential

   $$V(x,y,z) = \begin{cases} 0 & 0 < x,y,z < a \\ \infty & \text{otherwise} \end{cases} \quad . \tag{1}$$

   That is, the particle moves freely within a box with walls at $x = 0, a$, $y = 0, a$, and $z = 0, a$.

   (a) Find the wavefunctions and energies of the stationary states.

   (b) In 1D quantum mechanics, there is only one bound state for a given energy. In 3D, there can be more than one; we call stationary states with the same energy *degenerate*, and the number of states with a given energy is the *degeneracy*. Give the three lowest energy eigenvalues and their degeneracies.

   (c) Write the three lowest energy eigenvalues for a similar potential but with walls at $x = 0, 2a$, $y = 0, a$, and $z = 0, a$. What are the degeneracies?

2. **Isotropic Harmonic Oscillator** *from Griffiths 3.38,39*

   Now we consider a harmonic oscillator where the restoring force is independent of the direction. In this case, the potential is

   $$V(r) = \frac{1}{2} m\omega^2 r^2 \ . \tag{2}$$

   (a) Show that the energy eigenvalues are $E_n = \hbar\omega(n + 3/2)$, where n is any non-negative integer. It's easiest to do this using separation of variables in Cartesian coordinates.

   (b) Find the degeneracy of states with energy E_n.

   (c) Show that the (unnormalized) wavefunction

   $$R(r) = r \exp\left[-\frac{m\omega}{2\hbar} r^2\right] \tag{3}$$

   satisfies the radial equation for $\ell = 1$, $n = 1$.

3. **Electromagnetic Gauge Transformations** *Griffiths 4.61*

   Now that we're in 3D, we could imagine having an electromagnetic field. For a particle of charge q in potential Φ and vector potential $\vec{A}$, the Hamiltonian is

   $$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\Phi \ . \tag{4}$$

   The electric and magnetic field are

   $$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \ , \quad \vec{B} = \vec{\nabla} \times \vec{A} \ . \tag{5}$$

   For more details, see Griffiths problem 4.59.

(a) Show that the electromagnetic fields are invariant under *gauge transformations*. That is, show that the potentials

$$\Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \ , \quad \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \tag{6}$$

give the same $\vec{E}$ and $\vec{B}$ fields as Φ and $\vec{A}$, where Λ is any function of $\vec{x}$ and t.

(b) Since the Hamiltonian involves the potentials, it looks like we can't just make a gauge transformation in the quantum theory. However, assuming that a wavefunction $\Psi(\vec{x}, t)$ solves the time-dependent Schrödinger equation for potentials Φ and $\vec{A}$, show that

$$\Psi' = e^{iq\Lambda/\hbar}\Psi \tag{7}$$

solves the time-dependent Schrödinger equation for the potentials Φ' and $\vec{A}'$ given in (6).

This gauge invariance is a critical feature of the quantum mechanical theory of electromagnetism with profound consequences. We may explore aspects of it again in assignments.