# PHYS-4601 Homework 5 Due 11 Oct 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **The Virial Theorem** *Based on Griffiths 3.31*

   Consider 3D quantum mechanics.

   (a) Using Ehrenfest's theorem, show that

   $$\frac{d}{dt}\langle \vec{x}\cdot\vec{p}\rangle = \left\langle \frac{\vec{p}^2}{m}\right\rangle - \left\langle \vec{x}\cdot\vec{\nabla}V(\vec{x})\right\rangle \ . \tag{1}$$

   (b) Show that the left-hand side of (1) vanishes in a stationary state to prove the *virial theorm*

   $$2\langle K\rangle = \left\langle \vec{x}\cdot\vec{\nabla}V(\vec{x})\right\rangle \ , \tag{2}$$

   where $K$ is the kinetic energy. (The virial theorem holds classically, also, though without the expectation values.)

   (c) Using the virial theorem, find the ratio of (the expectation value of) the kinetic energy to the potential energy for the harmonic oscillator potential $V \propto \vec{x}^2$ and for the Coulomb potential $V \propto 1/|\vec{x}|$. *Hint:* Remember from electromagnetism (or Newton's law of gravity) that $\vec{\nabla}(1/|\vec{x}|) = -\vec{x}/|\vec{x}|^3$.

2. **Gaussian Wavepacket Part II** *based on Griffiths 2.22*

   Here we return to the Gaussian wavepacket in 1D, here looking at the time evolution for the free particle Hamiltonian. We recall from the last assignment that the wavefunction (at some initial time) can be written as

   $$|\Psi(t=0)\rangle = \int_{-\infty}^{\infty} dx\ \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}|x\rangle = \int_{-\infty}^{\infty} dp\ \left(\frac{1}{2\pi a\hbar^2}\right)^{1/4} e^{-p^2/4a\hbar^2}|p\rangle \ . \tag{3}$$

   (a) Evolve this state in time. First write $\langle p|\Psi(t)\rangle$ and then show that

   $$\langle x|\Psi(t)\rangle = \frac{(2a/\pi)^{1/4}}{\sqrt{1+2i\hbar at/m}} e^{-ax^2/(1+2i\hbar at/m)} \ . \tag{4}$$

   *Hint:* You may want to use the trick of "completing the squares" to evaluate a Gaussian integral somewhere.

   (b) Find the probability density $|\langle x|\Psi(t)\rangle|^2$. Using the result from assignment 3 that $\langle x^2\rangle = 1/4a$ at $t=0$, find $\langle x^2\rangle$ at a later time $t$ by inspection of the probability density. Qualitatively explain what's happening to the wavefunction as time passes.

   (c) What's the momentum-space probability density $|\langle p|\Psi(t)\rangle|^2$? Does $\langle p^2\rangle$ change in time? Does this state continue to saturate the Heisenberg uncertainty relation for $t \neq 0$?

3. **Double Delta-Function Well** *based on Griffiths 2.27*

   Consider the potential

   $$V(x) = -\alpha\left[\delta(x+a) + \delta(x-a)\right] \ . \tag{5}$$

   As we did for the single delta-function well, define $\kappa = \sqrt{-2mE}/\hbar$.

(a) Since $V(x)$ is an even function, a stationary state wavefunction is either even or odd. Find the even bound state wavefunctions ($\psi(-x) = \psi(x)$) and a transcendental equation for $\kappa$. Don't bother normalizing the wavefunction. Give a graphical argument to count the number of allowed energies for even bound state wavefunctions (by "graphical," I mean show a sketch or computer-generated plot).

(b) Find the odd bound state wavefunctions ($\psi(-x) = -\psi(x)$) and a transcendental equation for $\kappa$. What condition must $\alpha$ and $a$ satisfy for an odd bound state to exist?