PHYS-4601 Homework 4 Due 4 Oct 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Normalization and Unitary Operators**

   Since the normalization condition $\langle\Psi|\Psi\rangle = 1$ represents the fact that all probabilities sum to one, it is very important that this normalization be time-independent in any physical system. We'll investigate this statement.

   (a) Consider two states $|\Psi(t)\rangle$ and $|\Phi(t)\rangle$ that both evolve according to the Schrödinger equation. Show that $\langle\Phi(t)|\Psi(t)\rangle$ is time-independent by differentiating and using the Schrödinger equation.

   (b) Now we'll take a different approach. First, consider *unitary* operators U, which are operators that satisfy $U^\dagger = U^{-1}$. If we transform our Hilbert space so that $|\Psi'\rangle = U|\Psi\rangle$ for all states $|\Psi\rangle$ and unitary U, show that $\langle\Phi'|\Psi'\rangle = \langle\Phi|\Psi\rangle$.

   (c) Show that $U = \exp[iA]$ is unitary if the operator A is Hermitian (define the exponential by its power series). *Hint:* You may want to show that $(AB)^\dagger = B^\dagger A^\dagger$. This shows that the time evolution operator $\exp[-iHt/\hbar]$ is unitary and therefore preserves inner products.

   (d) One last thing about unitary operators: show that if $U|\psi\rangle = \lambda|\psi\rangle$ (ie, $|\psi\rangle$ is an eigenstate of U), then $|\psi\rangle$ is an eigenstate of U^{-1} with eigenvalue $1/\lambda$. Use this fact to show that an eigenvalue λ of a unitary operator U satisfies $|\lambda|^2 = 1$.

2. **Heisenberg Equation**

   In the Schrödinger picture, states evolve in time according to the Schrödinger equation. In the Heisenberg picture, states are constant, but operators evolve in time. In this problem, you will find the differential equation that governs the evolution of an operator.

   Start by considering an operator with *no explicit time dependence*. In the Heisenberg picture, such an operator $\mathcal{O}$ satisfies

   $$\mathcal{O}(t) = e^{iHt/\hbar}\,\mathcal{O}(0)\,e^{-iHt/\hbar} \tag{1}$$

   for Hamiltonian H. Find the derivative $d\mathcal{O}/dt$ rigorously as follows:

   (a) First, by expanding the exponential as a power series and differentiating each term, find the time derivative of $\exp[\pm iHt/\hbar]$.

   (b) Then use the product rule to write $d\mathcal{O}/dt$ as a commutator. This is the Heisenberg equation (the equation $d\mathcal{O}/dt =$something).

   Two more short questions:

   (c) Is your Heisenberg equation consistent with Ehrenfest's theorem?

   (d) Consider the free particle Hamiltonian $H = p^2/2m$. Using the Heisenberg equation, find the operator $x(t)$ in terms of the operators $x(0)$ and $p(0)$.

3. **A Two-State System**

   Consider some physical system which only has two states, so its states $|\Psi\rangle$ can be represented by column vectors with two elements. In some basis, the Hamiltonian can be written as

   $$H \simeq \hbar\omega \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{2}$$

(a) Find the stationary states and corresponding energies.

(b) At time $t = 0$, the system is in state $|\Psi(0)\rangle \simeq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What is the smallest positive value of t such that $|\Psi(t)\rangle \propto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

(c) Show that the general time-evolved state is

$$|\Psi(t)\rangle \simeq \left(\cos(\omega t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - i \sin(\omega t) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) |\Psi(0)\rangle \tag{3}$$

for an initial state $|\Psi(0)\rangle$.