PHYS-4601 Homework 3 Due 27 Sept 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Some Commutator Relations**

   For operators A, B, C:

   (a) show that
   $$[A, BC] = [A, B]C + B[A, C] \ . \tag{1}$$

   (b) prove by induction that
   $$[A, B^n] = n[A, B]B^{n-1} \ , \tag{2}$$
   if $[A, B]$ commutes with B.

   Finally, consider the position and momentum operators, which have $[x, p] = i\hbar$.

   (c) Show using (2) that $[p, f(x)] = -i\hbar df/dx$. Assume $f(x)$ can be written as a Taylor series.

2. **Incompatible Operators** *Griffiths 3.15*

   We call any two operators A and B incompatible if $[A, B] \neq 0$. Show that two incompatible operators cannot share a complete basis of eigenstates. *Hint:* Go about this by contradiction; show that if there exists a complete basis of states that are eigenstates of both A and B, then the action of $[A, B]$ on any state vanishes.

3. **A Secret Look at Harmonic Oscillators**

   Consider some system (as we'll see in the weeks ahead, this could be a simple harmonic oscillator) where the orthonormalized energy eigenstates are labeled $|n\rangle$, where $n = 0, 1, 2, 3, \ldots$. Define the "lowering operator" a in terms of the dyad sum
   $$a = \sum_{n=1}^{\infty} \sqrt{n}|n-1\rangle\langle n| \ . \tag{3}$$

   (a) Find the Hermitian adjoint $a^\dagger$ of a and find the commutator $[a^\dagger, a]$. *Hint:* Remember how we do the adjoints of bras and kets. And also remember that Kronecker delta symbols allow you to eliminate summations.

   (b) Define the position and momentum operators as
   $$x = \sqrt{\hbar/2m\omega}\left(a^\dagger + a\right) \ , \quad p = i\sqrt{\hbar m\omega/2}\left(a^\dagger - a\right) \ . \tag{4}$$
   Show that the state $|0\rangle$ gives the minimum possible uncertainty $\sigma_x\sigma_p = \hbar/2$.

4. **Gaussian Wavepacket Part I**

   Here we take a first look at the Gaussian wavepacket in 1D, which is an important state in more than one physical system. In this problem, we will consider the state at a single instant $t = 0$, ignoring its time evolution. The state is
   $$|\psi\rangle = \int_{-\infty}^{\infty} dx \ Ae^{-ax^2}|x\rangle \ . \tag{5}$$

(a) Find the normalization constant A. *Hint:* To integrate a Gaussian, consider its square. When you square it, change the dummy integration variable to y, then change the integral over $dxdy$ to plane polar coordinates.

(b) Since the wavefunction is even, $\langle x \rangle = 0$. Find $\langle x^2 \rangle$. *Hint:* You can get a factor of x^2 next to the Guassian by differentiating it with respect to the parameter a.

(c) Write $|\psi\rangle$ in the momentum basis. *Hint:* If you have a quantity $ax^2 + bx$ somewhere, you may find it useful to write it as $a(x + b/2a)^2 - b^2/4a$ by completing the square. Then shift integration variables so it looks like you have a Gaussian again.

(d) Find $\langle p \rangle$ and $\langle p^2 \rangle$ and show that this state saturates the Heisenberg uncertainty principle.