PHYS-4601 Homework 2 Due 20 Sept 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Boundary Conditions and Operators**

   Consider a particle in 1D confined to the line segment $0 < x < L$ (note that the Hamiltonian is not specified). All wavefunctions must satisfy Dirichlet boundary conditions $\psi(0) = \psi(L) = 0$. It is easy to see that functions with these boundary conditions and the usual inner product

   $$\langle\psi|\phi\rangle = \int_0^L dx\, \psi^*(x)\phi(x) \tag{1}$$

   form a Hilbert space.

   (a) Show that momentum $p \simeq -i\hbar d/dx$ satisfies the Hermiticity condition $\langle\psi|p|\phi\rangle = (\langle\phi|p|\psi\rangle)^*$.

   (b) Argue briefly that any eigenstate of p must be an eigenstate of p^2.

   (c) Find the eigenstates and eigenvalues of the operator p^2 in this Hilbert space. Why aren't the eigenstates of p^2 also eigenstates of p? *Hint:* Ask if p acting on any wavefunction in this Hilbert space always gives another wavefunction in this Hilbert space.

   (d) Now change the boundary conditions to *Neumann* boundary conditions ($d\psi/dx = 0$ at $x = 0, L$). Does p satisfy the Hermiticity condition? Is p a linear operator on this new Hilbert space?

   The lesson of this problem is to be careful with naive assumptions; boundary conditions can have a nontrivial effect.

2. **Projectors and Dyad Operators**

   Start with a 3D Hilbert space with orthonormal basis $|1\rangle,\ |2\rangle,\ |3\rangle$. Consider 2 states

   $$|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle\ ,\quad |\beta\rangle = i|1\rangle + 2|3\rangle\ . \tag{2}$$

   (a) Show that

   $$|\alpha'\rangle = |\alpha\rangle - |\beta\rangle\langle\beta|\alpha\rangle/\langle\beta|\beta\rangle \tag{3}$$

   is orthogonal to β. This process of orthogonalizing vectors is called the Gram-Schmidt process and can be used to construct an orthonormal basis given enough linearly independent vectors.

   (b) *Griffiths problem 3.22(c)* Consider the operator $A = |\alpha\rangle\langle\beta|$. Is this Hermitean? Write A as a matrix in the orthonormal basis given.

3. **Functions of Operators**

   (a) Suppose $|\lambda\rangle$ is an eigenfunction of $\mathcal{O}$, $\mathcal{O}|\lambda\rangle = \lambda|\lambda\rangle$. For any function $f(x)$ that can be written as a power series

   $$f(x) = \sum_n f_n x^n\ , \tag{4}$$

   we can define

   $$f(\mathcal{O}) = \sum_n f_n \mathcal{O}^n\ . \tag{5}$$

Show that

$$f(\mathcal{O})|\lambda\rangle = f(\lambda)|\lambda\rangle \ . \tag{6}$$

Does this result hold if the power series includes negative powers?

(b) What are the eigenstates $|\lambda\rangle$ and eigenvalues of the operator $T_a = \exp[-ipa/\hbar]$? Assume the states are functions on $-\infty < x < \infty$ and give the most general form for an eigenstate $|\lambda\rangle$. *Hint:* Do different momentum eigenstates have the same eigenvalue?

(c) Show that T_a translates a wavefunction by a distance a. That is, show that

$$T_a|\psi\rangle \simeq \psi(x-a) \tag{7}$$

in the position basis. Show that $\langle x+a|\lambda\rangle$ is equal to a phase times $\langle x|\lambda\rangle$ for an eigenstate $|\lambda\rangle$ of T_a using the results of part (b). *Hint:* Think about the wavefunction's Taylor series around x.