PHYS-4601 Homework 17 Due 7 Mar 2013

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Some Sinusoidal Perturbations**

   In this problem, consider a Hamiltonian $H = H_0 + H_1(t)$, where we know the eigenstates and eigenvalues of H_0 and where $H_1(t)$ is a small time-dependent contribution to the Hamiltonian. We will consider a 2-state system where $|1\rangle$ and $|2\rangle$ are the eigenstates of H_0.

   (a) Consider a perturbation Hamiltonian

   $$\langle 1|H_1|1\rangle = \langle 2|H_1|2\rangle = 0 \ , \quad \langle 1|H_1|2\rangle = \langle 2|H_1|1\rangle = \begin{cases} 0 & t < 0 \text{ or } t > T \\ V & 0 \le t \le T \end{cases} \ . \tag{1}$$

   If the initial state is $|\Psi(t=0)\rangle = |1\rangle$, find the probability that a measurement finds the system in state $|2\rangle$ at time $t = T$ to first order in H_1. *Hint*: You may directly do the integration or use the limit of the sinusoidal perturbation discussed in class.

   (b) As in assignment 11 problem 1, consider a spin-1/2 particle (for example, a proton) with gyromagnetic ratio γ in the presence of a magnetic field

   $$\vec{B} = B_0\hat{z} + B_1\cos(\omega t)\hat{x} + B_1\sin(\omega t)\hat{y} \tag{2}$$

   at its fixed position. Show first that the Hamiltonian takes the form we're considering in this problem with the sinusoidal time-dependence discussed in the class notes. Then find the transition probability from spin up ($|1\rangle$) to spin down ($|2\rangle$) using your solution from assignment 11 problem 1(c) and show that it reduces to the perturbation theory result when $\gamma B_1 \ll \omega + \gamma B_0$.

2. **Fermi's Golden Rule**

   Consider a sinusoidal perturbation Hamiltonian $H_1 = Ve^{-i\omega t} + V^\dagger e^{+i\omega t}$. In the class notes, we found the probability for a transition from state $|1\rangle$ to $|2\rangle$ as a function of time and frequency ω. In the following, define $\hbar\omega_0 = E_2 - E_1$, the difference of the energy eigenvalues of the unperturbed Hamiltonian H_0. We will investigate the transition probability near $\omega = \omega_0$ at large t (at least as long as the probability stays small).

   (a) At a fixed (and large) time, the probability is peaked at $\omega = \omega_0$. Using L'Hospital's rule or just a power series expansion, find the peak transition probability as a function of time.

   (b) Find the values of ω where the probability first vanishes on either side of $\omega = \omega_0$. The difference in these two values tells us the width of the peak.

   (c) For large enough times, approximate the transition probability as a rectangle with the peak value from part (a) and width given by half the difference in part (b). Integrate this approximate probability function and argue that

   $$P \to \frac{2\pi|V_{21}|^2}{\hbar^2} t\delta(\omega_0 - \omega) \tag{3}$$

   as $t \to \infty$.

   This problem shows two things: first, transitions occur only to states at energies related by the perturbation frequency and, second, that there is a constant transition rate (probability per unit time) to the appropriate states. The relationship (3) is known as *Fermi's Golden Rule*.

3. **Anharmonic Oscillator Again**

In assignment 17 problem 4, you considered a particle moving in the 1D potential

$$V(x) = \frac{1}{2}m\omega^2 x^2 + gx^3 \tag{4}$$

and considered perturbations of the harmonic oscillator ground state $|0\rangle$. Using the variational method, show that the true ground state energy of this potential is unbounded below (that is, if I give you any real number, demonstrate that the ground state energy is less than that number). We say that this potential is unstable and has no ground state. *Hint*: Think about a simple trial wavefunction that approximates a delta function in position.