PHYS-4601 Homework 15 Due 14 Feb 2013

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Deformed Particle in a Box** *inspired by Griffiths 6.1*

   Consider a particle in a 1D infinite square well with boundaries at $x = \pm a$. If the particle is free inside the box $(-a < x < a)$, the wavefunctions and energies are

   $$\psi_n(x) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{a}} \cos\left(\frac{n\pi}{2a}x\right) & n \text{ odd} \\ \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi}{2a}x\right) & n \text{ even} \end{array} \right\} , \quad E_n^{free} = \frac{\hbar^2}{2m}\left(\frac{n\pi}{2a}\right)^2 . \tag{1}$$

   In the following, use first-order perturbation theory in the case that the particle is not free inside the box.

   (a) For potential (inside the box)

   $$V(x) = \left\{ \begin{array}{ll} V_0 & -a/2 < x < a/2 \\ 0 & |x| \geq a/2 \end{array} \right. , \tag{2}$$

   find the approximate ground state energy. Write this energy as E_1^{free} times a function of $\epsilon = ma^2V_0/\hbar^2$.

   (b) For potential (inside the box)

   $$V(x) = \alpha\delta(x) \tag{3}$$

   find the approximate energies for all the states. Show that the first-order corrections to the energy are small only when $ma\alpha/\hbar^2 \ll 1$.

2. **Changing Spring Constant** *Griffiths 6.2 extended*

   The harmonic oscillator potential is $V = (1/2)kx^2 \equiv (1/2)m\omega^2x^2$, and we've studied the states, energy levels, and ladder operators quite a bit before. Now consider a second harmonic oscillator with spring constant $k' = k(1+\epsilon)$. *Instructions:* In this problem, do not use the wavefunction or carry out any integrals. Use the ladder operators; they will save you a lot of work.

   (a) Suppose that $\epsilon \ll 1$. Write the energy eigenvalues of the second oscillator in terms of k and ϵ first as an exact expression and then as a power series in ϵ to second order.

   (b) Write the potential V' for the second harmonic oscillator approximately as the first harmonic oscillator potential plus a perturbation which is first order in ϵ. Using that expression, find the difference in energies between the two harmonic oscillators at first order in ϵ using perturbation theory and compare that to your answer from part (a).

   (c) Finally, consider the ground states of the two operators, which we can call $|0\rangle$ and $|0'\rangle$. To first order in ϵ, find the difference in these two states (which is itself an unnormalized ket).

3. **Stark Effect** *based on Griffiths 6.36*

   The presence of an external electric field $E_0\hat{z}$ shifts the energy levels of a hydrogen atom, which is called the Stark effect. Consider the hydrogen atom to be described by the Coulomb potential; the external electric field introduces a perturbation

   $$H_1 = eE_0z = eE_0r\cos\theta . \tag{4}$$

We have already seen on homework that the expectation value of this Hamiltonian in the ground state $n = 1$ vanishes, so there is no shift in the ground state energy. In this problem, we consider the degenerate perturbation theory of the $n = 2$ states. *As spin does not enter, do not consider it in this problem.*

(a) The four states $|2,0,0\rangle$, $|2,1,0\rangle$, and $|2,1,\pm 1\rangle$ are degenerate at 0th order. Label these states sequentially as $i = 1,2,3,4$. Show that the matrix elements $W_{ij} = \langle i|H_1|j\rangle$ form the matrix

$$W = -3aeE_0 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \tag{5}$$

where empty elements are zero and a is the Bohr radius. *Hint*: Note that L_z commutes with H_1, so only states with the same quantum number m can have nonzero matrix elements; this will save you quite a bit of work. Then use the angular wavefunctions to see that all the diagonal elements of W must vanish. Finally, use the explicit wavefunctions to evaluate the remaining matrix elements of W (there should only be one independent one left).

(b) Diagonalize this matrix to show that $|\pm\rangle = (1/\sqrt{2})(|2,0,0\rangle \pm |2,1,0\rangle)$ are eigenstates of W. Find the first order shift in energies of $|\pm\rangle$. *Hint*: You may still want that exam solution. Note that the corrected eigenstates may still have contributions from other values of the principal quantum numbers n, but that doesn't quite matter.

(c) Finally, show that the states $|\pm\rangle$ have a nonzero dipole moment $p_z = -e\langle z\rangle$ and calculate it. You should not need to do any more calculations; just use your answer from part (b).