PHYS-4601 Homework 14 Due 31 Jan 2013

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Specific Heat of the Free-Electron (Fermi) Gas**

   In this problem, you'll explore the thermal properties of the electrons in a simple metal. We'll work at low temperatures and assume that the chemical potential $\mu = E_F$, the Fermi energy. (This assumption does not affect the lowest order approximations that we'll use.)

   (a) Recall that the total number of electrons and total energy of the gas (on average) are

   $$N = \sum_{states} \frac{1}{\exp[(\epsilon - \mu)/T] + 1} \ , \quad U = \sum_{states} \frac{\epsilon}{\exp[(\epsilon - \mu)/T] + 1} \ . \tag{1}$$

   Replace the sum over states with an integral over wavevectors as discussed in class, where the number of states of wavevector magnitude k is

   $$V\Omega(k)dk = \frac{V}{\pi^2}k^2 dk \ . \tag{2}$$

   Write instead this *density of states* in terms of the single particle energy ϵ. In other words, find $\Omega(\epsilon)$ if $\Omega(\epsilon)d\epsilon$ is the number of single particle states with energy between ϵ and $\epsilon + d\epsilon$.

   (b) Now the number and energy densities can be written $n = \int_0^\infty d\epsilon \Omega(\epsilon) f(\epsilon, \mu, T)$ and $\rho = \int_0^\infty d\epsilon \epsilon \Omega(\epsilon) f(\epsilon, \mu, T)$, where f is the Fermi-Dirac function as used in (1). The specific heat is defined as $C_V = d\rho/dT$; show that this can be written as

   $$C_V = \int_0^\infty d\epsilon \, (\epsilon - \mu)\Omega(\epsilon)\frac{df}{dT} \tag{3}$$

   if the number density is held fixed ($dn/dT = 0$).

   (c) Notice that the Fermi-Dirac function f only has a nonzero derivative for $\epsilon \sim \mu$ when $T \to 0$, so you can approximate $\Omega(\epsilon) = \Omega(\mu)$ in (3). In the $T \to 0$ limit, show that

   $$C_V = \Omega(E_F) I_2 T \ , \ \text{where} \ \ I_2 = \int_{-\infty}^{\infty} dx \, \frac{x^2 e^x}{(e^x + 1)^2} \ . \tag{4}$$

   (d) Assuming $I_2 = \pi^2/3$, show that $C_V = \pi^2 nT/2E_F$.

2. **Magnetization of Single Spin**

   Consider a single spin of total spin s in a magnetic field with Hamiltonian

   $$H = -\gamma \vec{S} \cdot \vec{B} \tag{5}$$

   as usual. Recall that the magnetic moment of the spin is $\vec{\mu} = \gamma \vec{S}$. In this problem, use the canonical ensemble of statistical mechanics with temperature T and choose $\vec{B}$ to lie in the z direction.

   (a) Write the partition function for this system in terms of hyperbolic trig functions (you'll need to sum the geometric series and rearrange a bit).

(b) Show that the average magnetic moment of this spin is

$$\langle \mu_z \rangle = T \frac{d \ln Z}{dB_z} = \hbar\gamma \left[\frac{(s+1/2)\cosh[(s+1/2)\hbar\gamma B_z/T]}{\sinh[(s+1/2)\hbar\gamma B_z/T]} - \frac{\cosh(\hbar\gamma B_z/2T)}{2\sinh(\hbar\gamma B_z/2T)} \right] \equiv \hbar\gamma s B_s(\hbar\gamma B_z/T) \, , \tag{6}$$

where $B_s(x)$ is the so-called *Brillouin function*.

(c) The total magnetization of N uncoupled spins is $M_z = N\langle \mu_z \rangle$, where $\vec{\mu}$ is the magnetic moment of a single spin. Find the magnetic susceptibility $\chi = dM_z/dB_z$ as a function of temperature and give the high and low temperature limits.