PHYS-4601 Homework 12 Due 17 Jan 2013

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Multiple Particle Wavefunctions** *based on a problem in Ohanian*

   Consider two free spin 1/2 particles, which have single particle states $|\psi_1\rangle = |\vec{p}_1\rangle|+\rangle$ and $|\psi_2\rangle = |\vec{p}_2\rangle|-\rangle$. These states are factorized into spatial states (in this case, momentum eigenstates) and spin states (eigenstates of S_z).

   (a) Write the two-particle wavefunction if the two particles are distinguishable (say, particle 1 is a proton and particle 2 is an electron).

   (b) Now suppose that both particles are electrons, so they are indistinguishable. Write the two-particle state which is an eigenfunction of the total spin operator $\vec{S}^2$ with eigenvalue given by $s = 0$.

   (c) Keeping indistinguishable electrons, now write the two-particle wavefunction for with total spin eigenvalues $s = 1$, $m = 0$.

   (d) Finally, consider the case where the particles are indistinguishable but instead have spin 0, so there is no spin part of their states. Write the allowed two-particle state.

2. **3-Particle States** *a mix of Griffiths 5.7 and 5.33*

   Consider three particles, each of which is in one of the single-particle states $|\alpha\rangle$, $|\beta\rangle$, or $|\gamma\rangle$, which are orthonormal.

   (a) If the particles are bosons, write down the state where one particle is in each of $|\alpha\rangle$, $|\beta\rangle$, and $|\gamma\rangle$. *Hint*: This state must be symmetric under the exchange of *any* pair of the bosons.

   (b) How many linearly independent states can you form if the particles are fermions? Write down all the possible linearly independent states. *Hint*: Similarly to the above, these states must be antisymmetric under the exhange of *any* pair of the fermions.

3. **Estimating Helium Better** *Griffiths 5.11 clarified*

   In this problem, we will estimate the ground state energy of a helium atom. We will imagine that the electron repulsion is a correction to the attraction between the electrons and the nucleus.

   (a) Consider the states of a single electron around a helium nucleus (which has twice the charge of a proton). Argue that the "helium Bohr radius" $a_{\text{He}} = a/2$, where a is the usual Bohr radius, and that therefore the single-electron ground state wavefunction is given by

   $$\langle\vec{x}|n = 1, \ell = 0, m = 0\rangle = \sqrt{\frac{8}{\pi a^3}} e^{-2r/a} \ . \tag{1}$$

   Next assume that the two electron helium groundstate is $|n = 1, \ell = 0, m = 0\rangle_1 |n = 1, \ell = 0, m = 0\rangle_2 |s = 0, m_s = 0\rangle$, where the total spin state is the antisymmetric singlet. (The spatial wavefunction is given by Griffiths eqn [5.30].) Briefly argue that the energy of this state, in the absence of electron repulsion, is given by Griffiths eqn [5.31].

   (b) Now find $\langle|\vec{x}_1 - \vec{x}_2|^{-1}\rangle$ in this state, as follows:

1. Use the trick of setting the z axis for $\vec{x}_2$ along $\vec{x}_1$ and the law of cosines to see $|\vec{x}_1 - \vec{x}_2| = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta_2}$.
2. Do the angular integrals for $\vec{x}_2$, noting that

$$\int_0^\pi d\theta \sin\theta f(\cos\theta) = \int_{-1}^1 dx f(x) \ .$$

   Your result will have square roots of perfect squares, which are equal to absolute values. *Be careful of that!*
3. Carry out the r_2 integral in two parts, $0 < r_2 \leq r_1$ and $r_1 < r_2 < \infty$.
4. Now do the $\vec{x}_1$ integrals.

Hint: The "exponential integrals" formula in the back cover of Griffiths will be helpful.

(c) Estimate the change in the ground state energy due to the electron repulsion as

$$\Delta E = \frac{e^2}{4\pi\epsilon_0} \left\langle \frac{1}{|\vec{x}_1 - \vec{x}_2|} \right\rangle \ . \qquad (2)$$

Write ΔE in terms of the Bohr radius a and estimate its value in eV. Then add this to the energy from part (a) to get a rough estimate of the He ground state energy. *Hint*: Remember that the hydrogen ground state energy is $-\hbar^2/2ma^2 = -13.6$ eV.