PHYS-4601 Homework 11 Due 10 Jan 2013

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **MRI Spin Flips** *Inspired by a text by Horbatsch*

   In this problem, consider a charged spin-1/2 particle in an electric field that is constant along z and rotates in the $x-y$ plane:

   $$\vec{B} = B_1 \cos(\omega t)\hat{x} + B_1 \sin(\omega t)\hat{y} + B_0 \hat{z} \ . \tag{1}$$

   An MRI works by starting with $B_1 = 0$, so all the spins align in the lowest energy spin state. Then it flips the spins with the rotating field; if the spins have flipped effectively, their relaxation back to the ground state (after B_1 is turned off) generates a signal.

   (a) Write the spin state of the particle in terms of spin $\pm 1/2$ states as $|\Psi(t)\rangle = a(t)|+\rangle + b(t)|-\rangle$. Find the time-dependent Schrödinger equation for the spin in the magnetic field.

   (b) This equation is analytically solvable, but it takes a while. Find an analytic solution using Maple assuming that the particle is spin up at time $t = 0$. Include a copy of your Maple code.

   (c) Simplify (you'll probably have to do this yourself) your solution and show that

   $$\begin{aligned} a(t) &= e^{-i\omega t/2}\left[\cos\left(\alpha t/2\right) + i\frac{(\omega+\gamma B_0)}{\alpha}\sin\left(\alpha t/2\right)\right] \\ b(t) &= ie^{i\omega t/2}\frac{\gamma B_1}{\alpha}\sin\left(\alpha t/2\right) \end{aligned} \tag{2}$$

   with $\alpha = \sqrt{\gamma^2 B_1^2 + (\omega+\gamma B_0)^2}$. Verify that (2) solves the Schrödinger equation using the `odetest` function in Maple and plot the real and imaginary parts of $a(t), b(t)$ for $\omega = \gamma B_0 = 10\gamma B_1$ for $t = 0$ to $3/\omega$. Include a copy of your Maple code.

   (d) Under what condition on ω is it possible for the spin to flip entirely to spin down (that is, $a(t) = 0$, $|b(t)| = 1$) at some time t? Plot $a(t)$ and $b(t)$ over several periods of the solution for some set of parameters that allows a full spin flip. Include a copy of your plot.

2. **Hadron Spins** *Griffiths 4.35 plus*

   Quarks are elementary particles with spin 1/2, which we see in bound states called *hadrons*. Hadrons come in two varieties. In the following, assume that the quarks have zero orbital angular momentum.

   (a) *Mesons* are formed of a quark and antiquark (think of it as two quarks). What are the possible total spin quantum numbers of a meson?

   (b) *Baryons* are formed of three distinct quarks. What are the possible total spin quantum numbers? How many complete sets of states are there for each of those total spins?

3. **Spin Interactions**

   (a) Two spin 1/2 particles are fixed in position but have interacting spins. Their Hamiltonian is

   $$H = J\vec{S}^{(1)} \cdot \vec{S}^{(2)} \tag{3}$$

for some constant J. Here $S^{(i)}$ is the spin operator of the ith particle. Find the energy eigenvalues of this system, their degeneracies, and the corresponding eigenstates. *Hint*: You will want to work in terms of the total spin quantum numbers.

(b) The two spins have the same gyromagnetic ratio γ. In the presence of a magnetic field, the Hamiltonian becomes

$$H = J\vec{S}^{(1)} \cdot \vec{S}^{(2)} + \gamma\vec{B} \cdot \left(\vec{S}^{(1)} + \vec{S}^{(2)}\right) \ . \tag{4}$$

Now find the energy eigenvalues and their degeneracies. You may take $\vec{B}$ to lie along the z direction.

4. **Center of Mass Frame and Reduced Mass**

In class, we treat the hydrogen atom as if it is an electron moving around a stationary proton. Of course, that can't be, since it violates conservation of momentum. What happens, of course, is that the proton hardly moves in the center of mass rest frame. However, it turns out that we can always describe a system of two particles in terms of a single particle. In this problem, consider two particles of masses m_1 and m_2.

(a) In quantum mechanics, the kinetic energy is given by a Laplacian operator. Consider the 1D case for simplicity. Then the kinetic Hamiltonian is

$$H = \frac{1}{2m_1}\frac{d^2}{dx_1^2} + \frac{1}{2m_2}\frac{d^2}{dx_2^2} \ , \tag{5}$$

where x_1 is the first particle's position and x_2 is the second particle's position. Show that this kinetic Hamiltonian can be written as

$$H = \frac{1}{2M}\frac{d^2}{dX^2} + \frac{1}{2\mu}\frac{d^2}{dx^2} \ , \tag{6}$$

where $M = m_1 + m_2$ is the total mass, $\mu = m_1 m_2/M$ is the reduced mass, $X = (m_1x_1 + m_2x_2)/M$ is the center of mass position, and $x = x_1 - x_2$ is the relative position.

The proof is essentially the same for the 3D Laplacian, and we then set the center of mass momentum to zero by choice of reference frame. Therefore, when we study the hydrogen atom, we are really using the reduced mass of the electron, which is nearly the electron mass because the proton is so much heavier than the electron.

(b) Imagine an atom made of an electron and a positron (anti-electron); these are called positronium. Positronium atoms are exactly like hydrogen atoms (in terms of energy eigenvalues) except the proton mass is replaced by the positron mass (which is equal to the electron mass). Find the ground state energy of positronium.

5. **Quantum Earth** *almost Griffiths 4.17*

In this problem, treat the earth-sun system as an analog of the hydrogen atom. Let M be the mass of the sun and m the mass of the earth.

(a) By comparing the Newtonian gravitational potential to the Coulomb potential of the hydrogen atom, write down the gravitational Bohr radius a_g and the quantum mechanical earth-sun energy E_n in terms of M, m, and the Newton constant G.

(b) Compare the classical energy of a planet in a circular orbit of radius r to your formula E_n and show that $n = \sqrt{r/a_g}$. Estimate n for the earth. Let r be 1 astronomical unit. Just give one significant digit. *Hint*: You can look up all the astrophysical data you need at `http://pdg.lbl.gov/2012/astrophysics-cosmology/astro-cosmo.html` under "Astrophysical Constants and Parameters." It also helps to remember the virial theorem, which says that kinetic energy is $-1/2$ potential energy for an orbit in a $1/r$ potential.

(c) Show that the total orbital angular momentum quantum number ℓ is approximately n; that is, ℓ is close to its maximum allowed value.

6. **Hydrogen Ground State**

In this problem, the wavefunction is always that of the ground state $n = 0$, $\ell = 0$, $m = 0$ of the hydrogen atom. Give all answers in terms of pure numbers and the Bohr radius.

(a) *Griffiths 4.14* For what value of r are you most likely to find the electron between r and $r + dr$? Recall that the volume of each spherical shell changes with radius.

The next 2 parts are based on Griffiths 4.13

(b) Find $\langle x \rangle$, $\langle y \rangle$, and $\langle z \rangle$. Do not do any integration but rather argue based on the rotational symmetry of the wavefunction.

(c) Find $\langle r^2 \rangle$. Using your result and the rotational symmetry of the wavefunction, find $\langle z^2 \rangle$ (no additional integration allowed).