PHYS-4601 Homework 1 Due 13 Sept 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Vector Space on a Circle**

   In this question, we consider a one-dimensional space where all functions must obey the periodicity condition $f(x) = f(x + 2\pi R)$. These are functions on a circle of radius R.

   (a) Prove that the (complex) functions on this circle form a vector space.

   (b) Prove that
   $$\langle f|g\rangle \equiv \int_0^{2\pi R} dx\, f^*(x)g(x) \tag{1}$$
   is an inner product. You may assume that $f(x)$ and $g(x)$ are L^2 normalizable functions as defined in the lecture notes.

   (c) Using the fact that the complex exponentials $|e_n\rangle = e^{inx/R}/\sqrt{2\pi R}$ form an orthonormal basis, calculate the inner product of $f(x) = \cos^3(x/R)$ and $g(x) = \sin(3x/R)$. You are not allowed to do any integrals.

2. **Something We'll Call "Momentum"**

   In class, we defined states $|x\rangle$ corresponding to a particle at a well-defined position. Now define states $|p\rangle$ with wavefunctions
   $$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \ . \tag{2}$$
   For simplicity, stick to one dimension.

   (a) Show that $\langle p'|p\rangle = \delta(p - p')$. *Hint:* You may find the formula
   $$\delta(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikz} \tag{3}$$
   helpful.

   (b) Show that the wavefunction $\psi(x) = \langle x|\psi\rangle$ and "momentum-space wavefunction" $\tilde{\psi}(p) = \langle p|\psi\rangle$ for any state ψ are Fourier transforms, as defined in Griffiths equation [2.102] (up to factors of $\hbar$).

   (c) Consider a vector we call $|p \cdot \psi\rangle$ with momentum-space wavefunction $p\tilde{\psi}(p)$ made from a general state $|\psi\rangle$. Show that
   $$\langle x|p \cdot \psi\rangle = -i\hbar \frac{d\psi}{dx}(x) \ . \tag{4}$$

3. **Superposition of States**

   Suppose $|\psi\rangle$ and $|\phi\rangle$ are two normalized state vectors, and so is $|\alpha\rangle = A(3|\psi\rangle + 4|\phi\rangle)$.

   (a) Find the normalization constant A in the case that

      i. $\langle\psi|\phi\rangle = 0$.

      ii. $\langle\psi|\phi\rangle = i$.

      iii. $\langle\psi|\phi\rangle = e^{i\pi/6}$.

   (b) Now suppose that $\langle\psi|\phi\rangle = 0$ and define a new state $|\beta\rangle = B(3e^{i\theta}|\psi\rangle + 4e^{2i\theta}|\phi\rangle)$ for some angle θ. Find the normalization constant B and $|\langle\alpha|\beta\rangle|^2$.