4-Vectors

- The velocity transformation rule is too complicated.
  (And we didn't even look at something like acceleration)
  - Why do we care? Remember, physical laws must be <u>covariant</u>:
    + The L.H.S. of an equation must have the same transformation as the RHS.
    + To check that, we need well-organized sets of variables with simple transformation rules
  - We've seen an analogous case: <u>Rotations</u>
    + Remember that <u>vectors</u> all have the same rotation transformation rule

$$x^{i'} = R^{i'}{}_j x^j$$ w/ <u>Einstein summation convention</u>.

+ $x^{i'} = S'$ vector, $x^j = S$ frame vector, $R^{i'}{}_j =$ rotation matrix

For example, for a z-axis rotation

$$[R^{i'}{}_j] = \begin{array}{c|c} & j \rightarrow \\ \hline i' \downarrow & \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Recall how the sum = matrix multiplication

+ It's easy to check if equations are covariant b/c we know what's a vector or a <u>scalar</u> (a rotational invariant)

  Vectors: position, velocity, acceleration, momentum, force, ...
  Scalars: time, temperature, energy, mass, distance, ...

+ A couple of examples:

  Newton: $F^i = ma^i \rightarrow R^{i'}{}_j F^j = m R^{i'}{}_j a^j \rightarrow F^{i'} = ma^{i'}$ ✓

  Coulomb potential: $V(r) = q_1 q_2 / 4\pi\epsilon_0 r \leftarrow$ all scalars ✓

- Rotations are linear transformations on coordinates.
  So are the Lorentz transformations (boosts). We can make covariance under boosts easy to understand with this analogy.

- We can turn our coordinates into 4-vectors (index notation, this is standard, but the reading uses odd notation)

$$x^{\mu} = (ct, x, y, z) \quad \text{or} \quad x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$$

- We will use Greek indices $\mu, \nu, \dots$ for all spacetime coordinates and Latin $i, j, \dots$ for spatial directions only

- Lorentz boosts on 4-vectors
  + As in rotations, write a Lorentz transformation as matrix mult.

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} \quad \text{ie} \quad (S' \text{ frame}) = (\text{boost}) \cdot (S \text{ frame})$$

  + The boost can be represented as a matrix

$$\left[\Lambda^{\mu'}{}_{\nu}\right] = \begin{array}{c|c} & \nu \rightarrow \\ \mu' \downarrow & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \gamma & -\gamma v/c & & \\ -\gamma v/c & \gamma & & 0 \\ & & 1 & \\ & 0 & & 1 \end{bmatrix} \end{array}$$

Working out the sums, as usual:

$$ct' = \gamma(ct) - \left(\gamma \frac{v}{c}\right)(x), \quad x' = \gamma x - \left(\gamma \frac{v}{c}\right)(ct), \quad y' = y,\ z' = z$$

  + The analogy with rotations goes farther:

$$\gamma^2 - \gamma^2 \frac{v^2}{c^2} = \frac{1}{(1 - v^2/c^2)}(1 - v^2/c^2) = 1 \quad \text{like} \quad \cosh^2\theta - \sinh^2\theta = 1$$

We can define <u>rapidity</u> θ with $\cosh\theta = \gamma$, $\sinh\theta = \gamma v/c$

Turns Λ into a "hyperbolic rotation"

- A 4-vector contains a normal vector $x^{\mu} = (x^0, x^i)$ and rotations fit inside a general 4D Lorentz transformation:

$$\Lambda^{0'}{}_0 = 1, \quad \Lambda^{0'}{}_i = \Lambda^{i'}{}_0 = 0, \quad \Lambda^{i'}{}_j = R^{i'}{}_j$$

gives

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} \Rightarrow x^{0'} = x^0, \quad x^{i'} = R^{i'}{}_j x^j \quad \text{(can you see this?)}$$

- Not every vector is a position. Everything that rotates like a position is a vector. Similarly, <u>anything that Lorentz transforms like a spacetime position is a 4-vector.</u>

— Scalar Products

- For rotations, the dot product turns 2 vectors into a scalar (so we call it a <u>scalar product</u>).
  + The definition is $a^i b^i = a^1 b^1 + a^2 b^2 + a^3 b^3$
  + The fact that this is invariant means
    $$a^{i'} b^{i'} = R^{i'}{}_j a^j R^{i'}{}_k b^k = a^j b^j \Rightarrow R^{i'}{}_j R^{i'}{}_k = \delta_{jk} \Rightarrow R^T R = 1$$
  In other words, rotation matrices are <u>orthogonal</u>
- We already have something like a scalar product — the invariant interval
  + If we consider δx^μ as a 4-vector,
    $$\delta s^2 = -(\delta x^0)^2 + (\delta x^1)^2 + (\delta x^2)^2 + (\delta x^3)^2$$
  + To write this with index notation, we need to introduce the <u>metric</u>
    $$\delta s^2 = \eta_{\mu\nu} \delta x^\mu \delta x^\nu \quad \text{where} \quad [\eta_{\mu\nu}] = \begin{array}{c|c} & \nu \\ \hline \mu & \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \end{array}$$
  + Suppose you calculate δs^2 in another frame
    $$\delta s^2 = \eta_{\mu'\nu'} \delta x^{\mu'} \delta x^{\nu'} = (\eta_{\mu'\nu'} \Lambda^{\mu'}{}_\alpha \Lambda^{\nu'}{}_\beta) \delta x^\alpha \delta x^\beta \quad S \text{ frame}$$
  For this to be the same as in the S frame, we need
    $$\eta_{\mu'\nu'} \Lambda^{\mu'}{}_\alpha \Lambda^{\nu'}{}_\beta = \eta_{\alpha\beta} \Rightarrow \Lambda^T \eta \Lambda = \eta \quad (\ast)$$
  <u>Note</u>: In special relativity, the metric η is the same in every frame
  + We say Λ is a Lorentz transformation if it satisfies $(\ast)$
  This includes 3D rotations $[\Lambda] = \begin{bmatrix} 1 & \\ & R \end{bmatrix}$, etc.

- This is a scalar product for any 2 4-vectors
  + Frame S: $\eta_{\mu\nu} a^\mu b^\nu = \eta_{\mu'\nu'} a^{\mu'} b^{\nu'}$; frame S' $= a \cdot b$
  + You can take the scalar product of a vector with itself to <u>square</u> it
    $a^2 > 0$ spacelike, $a^2 = 0$ lightlike, $a^2 < 0$ timelike
  classifies 4-vectors.

- Index Up, Index Down

- There are other ways to define 4-vectors:
  + We know about upper indices a^μ. The Lorentz transformation is $a^{\mu'} = \Lambda^{\mu'}{}_\nu a^\nu$. Sometimes these are called contravariant 4-vectors
  + But we can also define a "covariant" 4-vector $a_\mu = \eta_{\mu\nu} a^\nu$
    By each component, $a_0 = -a^0$, $a_i = a^i$. These are lower indices
  + What's the Lorentz transformation?
    $$a_{\mu'} = \eta_{\mu'\nu'} a^{\nu'} = \eta_{\mu'\nu'} \Lambda^{\nu'}{}_\beta a^\beta$$
    From (*) $\eta_{\mu'\nu'} \Lambda^{\nu'}{}_\beta = \bar{\Lambda}_{\mu'}{}^\alpha \eta_{\alpha\beta}$ where $\bar{\Lambda} = (\Lambda^T)^{-1}$ as matrices.
    (As an exercise, work out the components of $\bar{\Lambda}$ for a boost along x.)
    So we have $a_{\mu'} = \bar{\Lambda}_{\mu'}{}^\alpha \eta_{\alpha\beta} a^\beta = \bar{\Lambda}_{\mu'}{}^\alpha a_\alpha$
  + Contravariant/upper indices transform by $\Lambda^{\mu'}{}_\nu$
    Covariant/lower indices transform by $\bar{\Lambda}_{\mu'}{}^\nu$
  + We lower indices with $\eta_{\mu\nu}$. To raise an index, we use $\eta^{\mu\nu}$ defined as the matrix inverse of $\eta_{\mu\nu}$
    $$[\eta^{\mu\nu}] = \begin{bmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \quad \text{and} \quad a^\mu = \eta^{\mu\nu} a_\nu$$

- Contractions
  + There are many ways to write the scalar product
    $$a \cdot b = \eta_{\mu\nu} a^\mu b^\nu = a_\nu b^\nu = a^\mu b_\mu = \eta^{\mu\nu} a_\mu b_\nu$$
  + When a raised index is summed up with an identical lowered index, we say they are contracted. You can only contract upper with lower.
  + Because upper + lower indices transform inversely, contracted indices are Lorentz invariant! The contracted indices effectively disappear (like contracted letters in a word).

- Tensors are objects with multiple indices where each index transforms like a (contravariant or covariant) 4-vector index:
  + Example: $T_{\mu'\nu'}{}^{\lambda'} = \bar{\Lambda}_{\mu'}{}^{\alpha} \bar{\Lambda}_{\nu'}{}^{\beta} \Lambda^{\lambda'}{}_{\gamma} T_{\alpha\beta}{}^{\gamma}$ + SEE MORE EXAMPLES BELOW!
  + The only tensor we have seen so far is the metric

    $\eta_{\mu'\nu'} = \bar{\Lambda}_{\mu'}{}^{\alpha} \bar{\Lambda}_{\nu'}{}^{\beta} \eta_{\alpha\beta}$, which is actually invariant.

    (In matrix notation, this is $\eta = \bar{\Lambda} \eta \bar{\Lambda}^T$, the same as $\eta = \Lambda^T \eta \Lambda$)
  + Another example is the Levi-Civitá tensor $\epsilon_{\mu\nu\lambda\rho}$

    This is $=0$ when any of μ, ν, λ, ρ are equal and $= \pm 1$ according to the sign of the permutation of the indices.

    That is $\epsilon_{0123} = +1$, swapping any pair of indices changes sign
  + Electromagnetic fields also take the form of tensors in relativity.

- Covariance of Equations

- Remember, allowed physical laws should have the same form in all inertial frames. That is, all terms should transform the same way (be covariant).
  + Lorentz transformations are determined solely by indices. An eqn. is correctly covariant if all un-contracted indices are the same in each term.
  + Examples

    $a^{\mu} f_{\mu\nu} q_{\lambda} = b_{\nu\lambda}$ ✓ $\quad u^{\alpha} u_{\beta} = p^{\mu}$ X ← unmatched indices

    $p^{\mu} + k^{\mu} = q^{\mu}$ ✓ $\quad a^{\nu} T^{\mu\mu} = b^{\nu}$ X ← can't contract 2 upper indices

    $a^{\mu} b^{\mu} q_{\mu} p^{\lambda} = d^{\mu\lambda}$ X ← what is contracted? make sure to use different indices for different sums.

- We will see equations of varying forms, but we will find it useful to work with equations of the form scalar = scalar. That way, you don't have to worry (as much) about changing reference frames.

MORE ON TENSORS.

+ You can contract tensor indices like 4-vector indices.

So

$T_{\mu\nu}{}^{\lambda} a_{\lambda} = S_{\mu\nu}$ is a tensor w/ 2 lowered indices

$T_{\mu\nu}{}^{\lambda} a^{\nu} = S_{\mu}{}^{\lambda}$ is a tensor w/ 2 indices

$T_{\mu\nu}{}^{\lambda} a^{\nu} b_{\lambda} = S_{\mu}$ is a 4-vector (covariant)

$T_{\mu\nu}{}^{\lambda} a^{\mu} b^{\nu} = S^{\lambda}$ is a 4-vector (contravariant)

$T_{\mu\nu}{}^{\lambda} a^{\mu} b^{\nu} d_{\lambda} = S$ is a scalar (invariant)

So is $T_{\mu\nu}{}^{\lambda} T_{\alpha\beta}{}^{\gamma} \eta^{\mu\alpha} \eta^{\nu\beta} \eta_{\lambda\gamma}$

+ You can raise + lower tensor indices, too.

$T_{\mu}{}^{\nu\lambda} = T_{\mu\alpha}{}^{\lambda} \eta^{\nu\alpha}$

$T^{\mu\nu}{}_{\lambda} = T_{\alpha\beta}{}^{\gamma} \eta^{\mu\alpha} \eta^{\nu\beta} \eta_{\lambda\gamma}$ etc.

Particle 4-Velocity

The idea is to write a 4-vector for velocity

- Notice that the proper time along the worldline of a particle is Lorentz invariant and spacetime position is a 4-vector
  - Specifically, $d\tau = dt\sqrt{1-\vec{u}^2/c^2}$ is Lorentz invariant.
  - Also, $dx^\mu = (c\,dt, d\vec{x})$ is a 4-vector
  - Now consider $U^\mu = dx^\mu/d\tau$ derivative w.r.t. proper time

    The "denominator" is Lorentz invariant, "numerator" is 4-vector.

    Therefore $U^{\mu'} = \frac{d}{d\tau}(\Lambda^{\mu'}{}_\nu x^\nu) = \Lambda^{\mu'}{}_\nu \frac{dx^\nu}{d\tau} = \Lambda^{\mu'}{}_\nu U^\nu$
  - U^μ is a 4-vector. We call it 4-velocity.

- The components are interesting in terms of the normal velocity
  - $$U^0 = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1-\vec{u}^2/c^2}} = c\gamma(\vec{u})$$
  - $$\vec{U} = \frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{x}}{dt} = \gamma(\vec{u})\,\vec{u}$$

    Note that $|\vec{U}| \to \infty$ as $|\vec{u}| \to c$
  - Inverting this relationship, the coordinate velocity is

    $$\vec{u}/c = \vec{U}/U^0$$

- Comments + properties:
  - $U^0 > 0$ for any sensible particle traveling into the future
  - U^μ is always timelike and has square

    $$U^2 = U_\mu U^\mu = -c^2\gamma^2 + \gamma^2\vec{u}^2 = -c^2\gamma^2(1-\vec{u}^2/c^2) = -c^2$$

    This is true for any 4-velocity of any particle
  - This is really only defined for massive particles (ie, not photons)