The <u>Invariant Interval</u>

Beware: notation different than text! Particularly signs.

- Define the <u>invariant interval</u> δs^2 between 2 events by

$$\delta s^2 = -c^2 \delta t^2 + \delta \vec{x}^2 \qquad \delta \vec{x}^2 = \delta x^2 + \delta y^2 + \delta z^2, \text{ etc.}$$

- This is like a Pythagorean distance but with a sign on the time part
- What do we mean invariant?

  It's the <u>same</u> when calculated in any inertial reference frame.

- Proof:

  What we want to show is

$$-c^2\delta t^2 + \delta x^2 + \delta y^2 + \delta z^2 = -c^2 \delta t'^2 + \delta x'^2 + \delta y'^2 + \delta z'^2$$

  for any frames S + S' (inertial)

  + Use standard configuration
  + We know $\delta y = \delta y'$, $\delta z = \delta z'$, so need

$$-c^2\delta t^2 + \delta x^2 = -c^2 \delta t'^2 + \delta x'^2$$

  + Start in S' and use Lorentz transformations

$$-c^2\delta t'^2 + \delta x'^2 = -c^2\gamma^2\left(\delta t - \frac{v\,\delta x}{c^2}\right)^2 + \gamma^2(\delta x - v\,\delta t)^2$$

$$= -c^2\delta t^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) + \delta x^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) + 2\gamma^2 v\,\delta t\,\delta x - 2\gamma^2 v\,\delta t\,\delta x$$

$$= -c^2\delta t^2 + \delta x^2 \quad \checkmark$$

- We have 3 cases
  - $\delta s^2 > 0$. Consider 2 events separated by $\delta\vec{x}$, δt with $c\,\delta t < |\delta\vec{x}|$.

    Then you can go to a frame with $\delta t' = \gamma\left(\delta t - \frac{v}{c^2}\delta x\right) = 0$

    How? Line up $\delta\vec{x}$ along x-axis. Choose $v = c^2\delta t/\delta x$.

    Then, in new frame,

$$\delta s^2 = (\delta x')^2$$

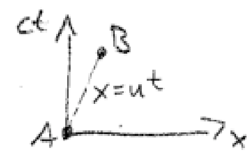
    Call this the (squared) <u>proper</u> <u>distance</u>. This is <u>spacelike</u> <u>separation</u>

  - $\delta s^2 < 0$.

- $\delta s^2 < 0$. This is time-like separation.
  We can choose a frame where $\delta \vec{x}' = 0$. So $\delta s^2 = -c^2 \delta t'^2$.
  We define $\delta\tau^2 = -\delta s^2/c^2 = +\delta t^2 - \delta\vec{x}^2/c^2 = (\delta t')^2$
  $\delta\tau$ is the proper time between the two events.
- $\delta s^2 = 0$. Imagine a lightray. Since light has $|\delta\vec{x}| = c\delta t$,
  $\delta s^2 = -c^2\delta t^2 + \delta\vec{x}^2 = -c^2\delta t^2 + c^2\delta t^2 = 0$. That means light can travel between the two events. They are light-like separated

— Examples

- Suppose A and B are timelike separated events. Then $|\delta\vec{x}/\delta t| < c$
  Some object moving at speed $u < c$ can go from A to B. Then

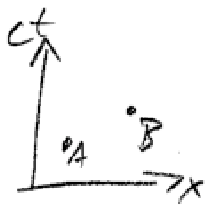


$$\delta s^2 = -c^2\delta t^2 + \delta\vec{x}^2 = -c^2\delta t^2 + u^2\delta t^2$$

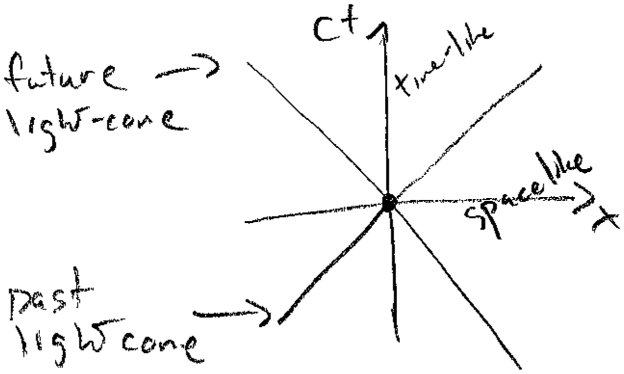
The proper time of the object's clocks is $\delta\tau = \sqrt{-\delta s^2/c^2} = \delta t\sqrt{1-u^2/c^2}$
Familiar?

- **Space time Diagrams**

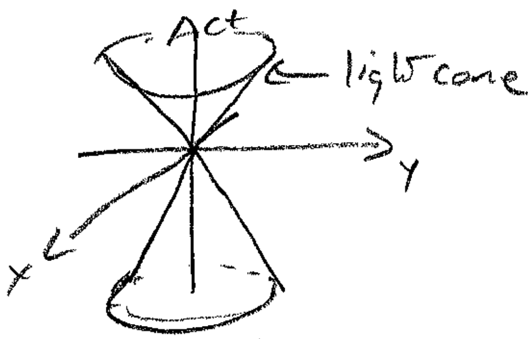
— For a while now, we've drawn diagrams as to the right indicating events as points. But we can do more!



— First, if we take the origin as one event, we can divide the diagram into time-like, space-like, and light-like regions

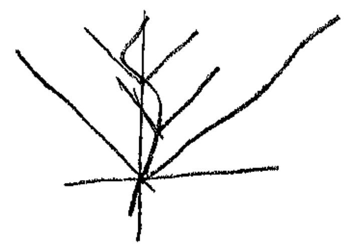


Note: light travels at 45°

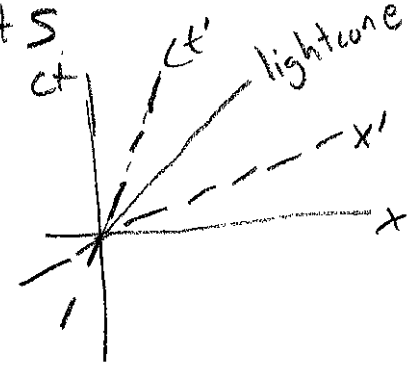


- The light-cone is all the light-like separated events.
- From an event, you can send a signal to anything on or in your future light cone
- And you can receive a signal from anywhere on or in your past lightcone
- No matter your frame, the light-cone is at 45°!

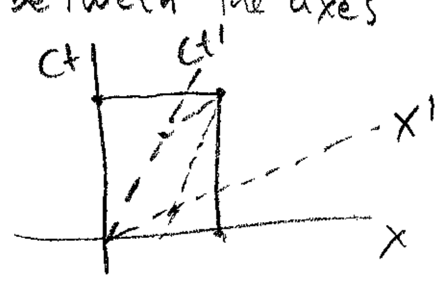
— We can draw the trajectories of objects/particles through time. These are called <u>worldlines</u>.
World lines can never bend out of the lightcone at any point
Useful for illustration



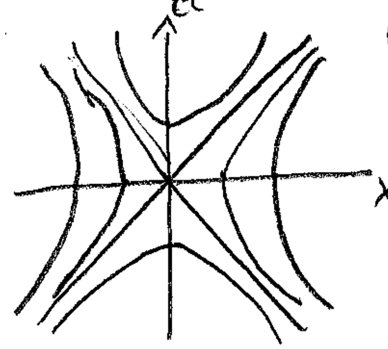
— Different frames in one diagram:
- Take S' in standard configuration w.r.t S
- What's the t' axis? That's just where $x'=0$, or $ct = x(c/v)$
- How about x' axis? That's where $t'=0$, or $ct = \left(\frac{v}{c}\right)x$



- The lightcone is still halfway between the axes
- To get coordinates, draw lines parallel to ~~x~~axes

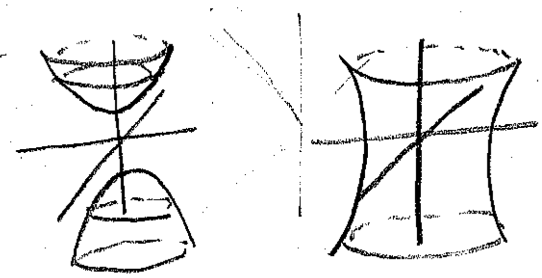


– Other invariant curves:



- Just like δs^2 is invariant, $c^2t^2 - \vec{x}^2$ is invt (From the same Lorentz transformations)
- Curves where this $=0$ are light cones
- Otherwise $=\pm(\#)$ are hyperboloids (hyperbolas in 2D)

Note that these rotate into y, z as well. like lightcones



- By a Lorentz transformation, you can move any point on a hyperboloid to any other point on the same hyperboloid
  - + Shows that space like separated events have no notion of "past", "future," or "simultaneity"! You can change time order
  - + But time like separated events <u>can't change order</u>! There is a notion of past and future, but only inside the light-cone.
- This shows us the <u>causal structure</u> of space time: which events can cause something to happen at other events.
  - + Note: That's why you can't travel faster than c. If you could get outside the light cone, you could change frames and "time-travel" to the past. Then you could kill your grandparents before your parents are born!