- The Boltzmann factor + Equilibrium Distributions
  - The relative probability (or population) of different states
    - For any particle (or point in single-particle phase space)

$$\frac{P(\vec{x}_1,\vec{p}_1)}{P(\vec{x}_2,\vec{p}_2)} = \frac{f(\vec{x}_1,\vec{p}_1)}{f(\vec{x}_2,\vec{p}_2)} = e^{-\Delta E/kT} \quad ; \quad \Delta E = E(\vec{x}_1,\vec{p}_1) - E(\vec{x}_2,\vec{p}_2)$$

↑ Boltzman factor

    - So, suppose you are looking at particles in gravity.

This means $f(y)/f(0) = e^{-mgy/kT}$ controls density of air as function of height

    - Where does this come from? Let's imagine 1 air molecule interacting with the rest of the molecules in the room

      + The total energy should be conserved. So, if an molecule has energy E, the rest of the room should have

$$E_{rest} = E_{tot} - E$$

      + For each state (point in phase space), the probability for the single particle to occupy it is $\propto$ # of states of the rest of the molecules

$$P(\vec{x},\vec{p}) \propto \Omega(E_{tot} - E) = \exp\left[\ln\Omega(E_{tot}) - E\,\frac{d\ln\Omega}{dE}(E_{tot})\right]$$

$$\propto \exp[-E/kT].$$

This is really just a "number counting" or combinatorical probability.

The probability of filling a state $\propto$ the # of ways of filling it

  - Major Example: Maxwell Velocity Distribution
    - Consider a gas of non-interacting particles (ignoring gravity + any other potential energy).
      + We'll ignore position + just look at the momentum distribution

+ Total energy of a particle is $E = KE = \vec{p}^2/2m$

+ Therefore, $f(\vec{p}) \propto e^{-\vec{p}^2/2mkT}$ or $f(\vec{v}) \propto e^{-m\vec{v}^2/2kT}$

by using the Boltzmann factor.

+ To normalize this, we note that

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \left[\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} \int_{-\infty}^{\infty} dy\, e^{-\alpha y^2}\right]^{1/2} = \left[\int_0^{2\pi} d\theta \int_0^{\infty} dr\, r\, e^{-\alpha r^2}\right]^{1/2}$$

$$= \left[2\pi \left(\frac{-1}{2\alpha} e^{-\alpha r^2}\Big|_0^{\infty}\right)\right]^{1/2} = \sqrt{\frac{\pi}{\alpha}}$$

Then, since $f(\vec{v})$ has three Gaussian integrals,

$$f(\vec{v}) = N \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} \qquad \left(\begin{array}{c}\text{Maxwell or} \\ \text{Maxwell-Boltzmann}\end{array}\right)$$

Usually, this is normalized so N = number/volume

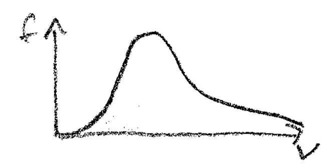
- Reduced Distributions

  + You can make a distribution for each component of velocity

$$f(v_x) = \int dv_y\, dv_z\, f(\vec{v}) = N \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-mv_x^2/2kT}, \text{ etc}$$

  + Or, in spherical coordinates, a distribution for the speed

$$\left.\begin{array}{l} \int dv\, f(v) = \int d^3\vec{v}\, f(\vec{v}) = \int dv\, d^2\Omega\, v^2 f(\vec{v}) \\ \Rightarrow \quad f(v) = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} \end{array}\right\} \text{ for } v > 0$$



  + An alternate derivation: $f(\vec{x}, \vec{p}) \propto$ (# of states) $\times$ Boltzmann factor

Then

$$\int dv\, f(v) \propto \int dE\, \Omega(E)\, e^{-E/kT} \propto \int dE \sqrt{E}\, e^{-E/kT}$$

Use $E = \frac{1}{2}mv^2$, $dE = mv\,dv$, so

$$\int dv\, f(v) \propto \int dv\, v^2 e^{-mv^2/2kT} \text{ as before}$$

- Averages
  - The average of any velocity component $\langle v_x \rangle = \int_{-\infty}^{\infty} dv_x \, v_x P(v_x) = 0$
  - But the square is not

$$\langle v_x^2 \rangle = \int_{-\infty}^{\infty} dv_x \, v_x^2 P(v_x) = \left[\sqrt{\frac{\alpha}{\pi}} \left(-\frac{d}{d\alpha} \sqrt{\frac{\pi}{\alpha}} \right) \right] \Bigg|_{\alpha = \frac{m}{2kT}} = \frac{kT}{m}$$

  - The average speed is

$$\langle v \rangle = \int_0^{\infty} dv \, v \, P(v) = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \int_0^{\infty} dv \, v^3 e^{-mv^2/2kT}$$

Note $$\int_0^{\infty} dx \, x^3 e^{-\alpha x^2} = -\frac{d}{d\alpha} \int_0^{\infty} dx \, x e^{-\alpha x^2} = -\frac{d}{d\alpha} \left[+\frac{1}{2\alpha} \right] = \frac{1}{2\alpha^2}$$

Therefore $$\langle v \rangle = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \frac{2k^2T^2}{m^2} = \sqrt{\frac{8}{\pi}} \sqrt{\frac{kT}{m}}$$

  - And the avg. square speed is

$$\langle v^2 \rangle = \int_0^{\infty} dv \, v^2 P(v) = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \int_0^{\infty} dv \, v^4 e^{-mv^2/2kT}$$

Similar tricks... $\langle v^2 \rangle = 3kT/m = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle$

  - Notice: In a distribution, $\langle v^2 \rangle \neq \langle v \rangle^2$. This is very important and generally true!