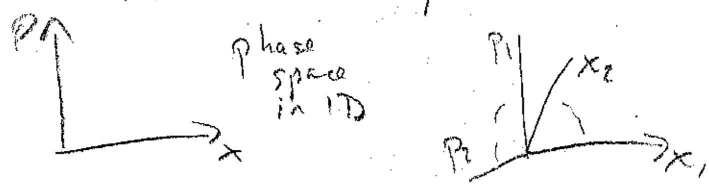
Statistical Distributions

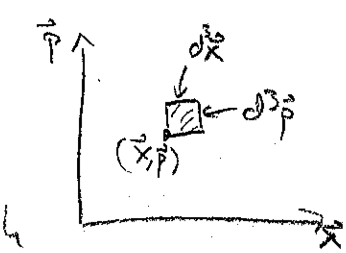
- We want a way to keep track of many particles.
  We can look at the distribution of particles through their states.

– Phase Space: How we define the state of a system in classical mech.

- Consider a particle moving around. + You know everything about its motion from position $\vec{x}$ and momentum $\vec{p}$.
  + We call the 6D space $(\vec{x}, \vec{p})$ phase space. (Generalizes to other dimensions easily).
  + You can clearly replace $\vec{p} \to \vec{v}$ if you want
  + For a system of particles, phase space is "bigger".
  You need $(\vec{x}, \vec{p})$ for each particle $\longrightarrow$ $(\vec{x}_1, \dots \vec{x}_N, \vec{p}_1, \dots \vec{p}_N)$



- How many particles of a system are at a given point of (single-particle) phase space?
  + Break phase space into infinitesimal blocks



  + If we consider the phase space position of each particle, the distribution function $f(\vec{x}, \vec{p})$ tells us

$$f(\vec{x}, \vec{p})\, d^3\vec{x}\, d^3\vec{p} = \# \text{ of particles in volume of phase space}$$

  + A distribution can be an average over microscopic times as particles move around (like air molecules in a room) or else it could be the exact time-dependent distribution

$$f(\vec{x}, \vec{p})\big|_{avg} \quad \text{vs} \quad f(\vec{x}, \vec{p}, t)$$

Let's focus on the average "equilibrium" version

— Averages, etc.

- The probability of any particle being in a given phase space volume is
$$P(\vec{x},\vec{p})\, d^3\vec{x}\, d^3\vec{p} = \tfrac{1}{N} f(\vec{x},\vec{p})\, d^3\vec{x}\, d^3\vec{p}, \quad N = \text{total \# of particles.}$$
  + Any physical observable that you might call $\mathcal{O}$ (like energy) is a function of phase space position $\mathcal{O} = \mathcal{O}(\vec{x},\vec{p})$.
  + Therefore, the average of the quantity is
$$\langle \mathcal{O} \rangle = \int d^3\vec{x} \int d^3\vec{p}\ \mathcal{O}(\vec{x},\vec{p})\, P(\vec{x},\vec{p})$$
  + An example: Particles uniformly distributed in a 1D "box" of length "L". The distribution is Gaussian in momentum
$$f(x,p) = \frac{N}{L}\left(\sqrt{\frac{a}{\pi}}\, e^{-ap^2}\right) \quad \text{this will be like an ideal gas}$$
  Then the average kinetic energy is $(K = p^2/2m)$
$$\langle K \rangle = \frac{1}{2m}\sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} dp\ p^2 e^{-ap^2} = \frac{1}{2m}\sqrt{\frac{a}{\pi}}\left(-\frac{d}{da}\sqrt{\frac{\pi}{a}}\right) = \frac{1}{4ma}$$
  We'll see the meaning of the constant a later.

- You can also have a distribution over a reduced phase space.
  + As in our example, the distribution is uniform in x. Therefore, most of the physical content is in the "reduced" distribution over p only: $f(p) = N\sqrt{\frac{a}{\pi}}\, e^{-ap^2}$
  + Pragmatically, we get the reduced distribution by integrating over the variables we don't want. Suppose $\{q\}$ is the subset of $(\vec{x},\vec{p})$ we want and $\{y\}$ are the variables we don't. $\Longrightarrow$
$$f(q) = \int d^n y\ f(q,y)$$
  + Another example: angular distribution. Ignore position and write $\vec{p}$ in spherical coordinates (p,θ,ϕ).
$$f(\vec{p})\, d^3\vec{p} = f(p,\theta,\phi)\, p^2\, dp\, d^2\Omega \qquad d^2\Omega = \sin\theta\, d\theta\, d\phi = \text{solid angle}$$

The angular distribution is $f(\theta,\phi) = \int dp\, p^2 f(p,\theta,\phi)$.

A uniform angular distribution is $f(\theta,\phi) = N/4\pi$ ← a sphere has 4π steradians

- Density of States:

• We might want to think about the number of states available at a given energy, since that's an important physical quantity

+ The <u>density of states</u> $\Omega(E)$ is # states per energy (also written as $\Omega(E)\,dE$ = # states w/ energy E to $E+dE$)

+ In classical mechanics, this is proportional to the volume of phase space that has the appropriate energy.

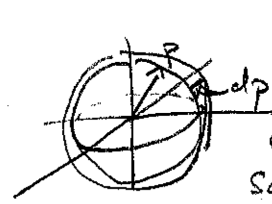
• An <u>example</u>: + Let's find $\Omega(E)$ for a free particle (no potential energy) in a volume V. We know

$$\int \Omega(E)\,dE \propto \int d^3\vec{x}\, d^3\vec{p} = V \int p^2 \sin\theta\, dp\, d\theta\, d\phi = 4\pi V \int dp\, p^2$$

Now use $p^2/2m = E$, $dE = p\frac{dp}{m}$. Then

$$\int \Omega(E)\,dE \propto V \int m\, dE \sqrt{2mE}$$

$$\Longrightarrow \Omega(E) \propto V\sqrt{E}$$



surface area of sphere

+ The picture is much the same for many particles, except you need the surface area of a many-dimensional sphere

$$\Omega(E) \propto E^{\text{large number} \sim N = \#\text{ particles}}$$

• Mostly you should just recognize this concept. But we will see it once or twice more this term

+ Also note that Ω increases <u>fast</u> as E increases for many particles

We can see that

$$\ln \Omega \propto N \ln E \Rightarrow \frac{d\ln\Omega}{dE} \propto \frac{N}{E}$$

E/N looks like energy/particle $\equiv kT$ = (Boltzmann's constant) $\times$ (temperature)

Use that as a definition