PHYS-3301 Homework 8 Due 7 Nov 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Three Short Practice Calculations**

   Calculate the following quantities. You should get a number for each answer.

   (a) $\eta_{\mu\nu}\eta^{\mu\nu}$

   (b) $\eta^{\mu\nu}\eta^{\lambda\rho}\epsilon_{\mu\nu\lambda\rho}$

   (c) $\epsilon_{\mu\nu\lambda\rho}\epsilon^{\mu\nu\lambda\rho}$

2. **Derivatives Have Lowered Indices**

   As discussed in the class notes, 4-vectors with raised or lowered indices have the following Lorentz transformations:

   $$a^{\mu'} = \Lambda^{\mu'}{}_{\nu}a^{\nu} \quad \text{and} \quad a_{\mu'} = \bar{\Lambda}_{\mu'}{}^{\nu}a_{\nu} \ , \tag{1}$$

   where Λ is the usual Lorentz transformation matrix and $\bar{\Lambda}^T = \Lambda^{-1}$ as a matrix.

   (a) Show that the matrix relationship between $\bar{\Lambda}$ and Λ may be written as $\bar{\Lambda}_{\mu'}{}^{\rho}\Lambda^{\nu'}{}_{\rho} = \delta^{\nu'}_{\mu'}$ and $\bar{\Lambda}_{\rho'}{}^{\mu}\Lambda^{\rho'}{}_{\nu} = \delta^{\mu}_{\nu}$, where $\delta^{\nu'}_{\mu'}$ and δ^{μ}_{ν} are Kronecker delta symbols.

   (b) Using the fact that the spacetime position x^{μ} is a 4-vector, find the partial derivatives $\partial x^{\mu}/\partial x^{\nu'}$ and $\partial x^{\mu'}/\partial x^{\nu}$ in terms of $\Lambda^{\mu'}{}_{\nu}$ and $\bar{\Lambda}_{\mu'}{}^{\nu}$. *Hint:* For two positions as measured in the same frame, $\partial x^{\mu}/\partial x^{\nu} = \delta^{\mu}_{\nu}$ (think about why).

   (c) If f is a Lorentz invariant function (meaning its value at a fixed spacetime point is the same in any frame — like the temperature), use the chain rule to show that

   $$\frac{\partial f}{\partial x^{\mu'}} = \bar{\Lambda}_{\mu'}{}^{\nu}\frac{\partial f}{\partial x^{\nu}} \ . \tag{2}$$

   In other words, you are showing that a partial derivative has the same transformation as a 4-vector with a lowered index. As a result, people will usually write $\partial_{\mu}f \equiv \partial f/\partial x^{\mu}$.

3. **The Relativistic Electromagnetic Field**

   We won't prove it, but the electric and magnetic fields can be written as a relativistic tensor with two indices $F^{\mu\nu}$. This tensor is *antisymmetric*, meaning $F^{\nu\mu} = -F^{\mu\nu}$. The independent components are (here, $i = 1, 2, 3$ is a space index)

   $$F^{oi} = E^i \ , \quad F^{12} = B^3 \ , \quad F^{13} = -B^2 \ , \quad F^{23} = B^1 \ . \tag{3}$$

   Since $F^{\mu\nu}$ is antisymmetric, the diagonal components $F^{00} = F^{11} = F^{22} = F^{33} = 0$. (We have chosen a convenient system of units where the electric and magnetic field have the same dimension.)

   (a) Consider two frames S and S' in standard configuration with each other. Show that

   $$E^{3'} = \gamma\left(E^3 + \frac{v}{c}B^2\right) \quad \text{and} \quad B^{3'} = \gamma\left(B^3 - \frac{v}{c}E^2\right) \ . \tag{4}$$

   Hint: Remember that the Lorentz transformation of a tensor transforms each index independently:

   $$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\alpha}\Lambda^{\nu'}{}_{\beta}F^{\alpha\beta} \ . \tag{5}$$

   (b) Calculate $F_{\mu\nu}F^{\mu\nu}$ and argue that $\vec{E}^2 - \vec{B}^2$ is a Lorentz invariant quantity.