PHYS-3301 Homework 7 Due 31 Oct 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **A Short Calculation** *inspired by a problem in Hartle*

   In some frame, the components of two 4-vectors are

   $$a^\mu = (2,0,0,1) \text{ and } b^\mu = (5,4,3,0)\ . \tag{1}$$

   (a) Find a^2, b^2, and $a \cdot b$.

   (b) Does there exist another inertial frame in which the components of a^μ are $(1,0,0,1)$? What about b^μ?

2. **4-Vectors and Changing Frames**

   In parts (a)-(c), a^μ and b^μ are both timelike.

   (a) Show that there exists an inertial frame S where the only nonzero component is a^0 (that is, the spatial part $\vec{a}$ is zero).

   (b) Without using explicit Lorentz transformations, show that $|a^0|$ is minimized in the frame S defined in part (a).

   (c) *from the text by Hartle* Show that $a \cdot b = -\sqrt{a^2 b^2}\gamma$, where γ is the relativistic γ factor for the Lorentz transformation between the frame S where the spatial part of a^μ is zero and the frame S' where the spatial part of b^μ is zero.

   Now consider lightlike 4-vectors a^μ and b^μ.

   (d) Suppose the components of a^μ in frame S are $a^0 = k$, $a^1 = k$, and $a^2 = a^3 = 0$. If frame S' is related to S by a boost of velocity v along the x direction, show that the components in frame S' are $a^{0'} = k'$, $a^{1'} = k'$, and $a^{2'} = a^{3'} = 0$, where $k' = \sqrt{(1-v/c)/(1+v/c)}\, k$.

   (e) Is the 4-vector $a^\mu + b^\mu$ spacelike, timelike, or lightlike?

3. **Boosts and Rotations**

   In matrix form, we can define the boost Λ_{tx} along x and the rotation Λ_{xy} in the xy plane (around the z axis) as follows:

   $$\Lambda_{tx}(\phi) = \begin{bmatrix} \cosh\phi & -\sinh\phi & & \\ -\sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\ ,\quad \Lambda_{xy}(\theta) = \begin{bmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{bmatrix}\ . \tag{2}$$

   Empty elements in the matrices above are zero.

   (a) In matrix form, the metric $\eta_{\mu\nu}$ is

   $$\eta = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\ . \tag{3}$$

   Show that both rotation and boost in (2) satisfy the condition $\eta_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta}$, which is $\eta = \Lambda^T \eta \Lambda$ in matrix notation.

(b) Consider two successive boosts along x, $\Lambda_{tx}(\phi_1)$ and $\Lambda_{tx}(\phi_2)$. Show that these multiply to give a third boost $\Lambda_{tx}(\phi_3)$ and find ϕ_3. Using the relationship $v/c = \tanh\phi$ between velocity and rapidity ϕ, reproduce the velocity addition rule. *Hint:* You will need the angle-addition rules for hyperbolic trig functions.

(c) First, write down the Lorentz transformation matrix $\Lambda_{ty}(\phi)$ corresponding to a boost along the y direction by permuting axes. Then show that you can get a boost along y by rotating axes, boosting along x, then rotating back by proving that $\Lambda_{ty}(\phi) = \Lambda_{xy}(-\pi/2)\Lambda_{tx}(\phi)\Lambda_{xy}(\pi/2)$.