- **Time Dependence** Note: Not in text much

- Start with the general Schrödinger equation.

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle$$

- Can write this in position basis as

$$i\hbar \frac{\partial}{\partial t}\langle x|\Psi(t)\rangle = i\hbar\frac{\partial \Psi}{\partial t} = \langle x|H|\Psi(t)\rangle = -\frac{\hbar^2}{2m}\nabla^2\Psi + V(\vec{x})\Psi$$

- Or maybe momentum basis

$$i\hbar\frac{\partial}{\partial t}\langle p|\Psi(t)\rangle = \langle p|H|\Psi(t)\rangle = \frac{p^2}{2m}\tilde{\Psi} + V(i\hbar\vec{\nabla}_p)\tilde{\Psi}$$

- Or the energy basis

$$i\hbar\frac{d}{dt}\langle E_n|\Psi(t)\rangle = \langle E_n|H|\Psi(t)\rangle = E_n\langle E_n|\Psi(t)\rangle$$

+ This says we break $|\Psi(t)\rangle$ into components in energy basis
+ Each of those components has exponential time dependence
  Let $C_n(t) = \langle E_n|\Psi(t)\rangle$. Then $C_n(t) = C_n e^{-iE_nt/\hbar}$
+ Then linear decomposition gives usual "time independent" Schrödinger equation

$$|\Psi(t)\rangle = \sum_n C_n(t)|E_n\rangle$$

$$\langle x|H|E_n\rangle = -\frac{\hbar^2}{2m}\nabla^2\psi_n(\vec{x}) + V(\vec{x})\psi_n(\vec{x})$$

- Using functions of operators suggests something in the energy basis.

+ $$|\Psi(t)\rangle = \sum_n C_n e^{-iE_nt/\hbar}|E_n\rangle = \sum_n C_n e^{-iHt/\hbar}|E_n\rangle$$
  $$= e^{-iHt/\hbar}|\Psi(0)\rangle.$$

+ The time dependent solution looks like a <u>time-evolution operator</u> (aka <u>propagator</u>) acting on the initial wavefunction.

+ And $i\hbar\frac{d}{dt}\left(e^{-iHt/\hbar}|\Psi(0)\rangle\right) = He^{-iHt/\hbar}|\Psi(0)\rangle$. Interesting.

- Ehrenfest's Theorem: Expectation values obey classical physics
- Example: $\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}$, $\frac{d\langle p\rangle}{dt} = -\left\langle \frac{\partial V}{\partial x}\right\rangle$
- What's the time dependence of an expectation value?

$$\langle \mathcal{O}\rangle(t) = \langle \Psi(t)|\,\mathcal{O}(t)\,|\Psi(t)\rangle,$$ with explicit operator time-dep.

+ Recall Schrödinger eqn.

$$\frac{d}{dt}|\Psi(t)\rangle = \frac{-i}{\hbar}H|\Psi(t)\rangle \; ; \; \frac{d}{dt}\langle\Psi(t)| = \frac{i}{\hbar}\langle\Psi(t)|H$$

+ Then

$$\frac{d}{dt}\langle\mathcal{O}\rangle = \frac{i}{\hbar}\langle\Psi|H\mathcal{O}|\Psi\rangle + \langle\Psi|\frac{\partial\mathcal{O}}{\partial t}|\Psi\rangle - \frac{i}{\hbar}\langle\Psi|\mathcal{O}H|\Psi\rangle$$

$$= \frac{i}{\hbar}\langle[H,\mathcal{O}]\rangle + \left\langle\frac{\partial\mathcal{O}}{\partial t}\right\rangle$$

+ Time dependence is both explicit and implicit (through commutator)

- Let's consider some 1D: $H = \frac{p^2}{2m} + V(x)$

+ Then $[H,x] = [p^2,x]/2m = -\frac{1}{m}[x,p]p = -\frac{i\hbar}{m}p$

x has no explicit time dependence, so

$$\frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\left(-\frac{i\hbar}{m}\right)\langle p\rangle = \frac{\langle p\rangle}{m}$$ as predicted

+ And

$$[H,p] = -[p,V(x)] = +i\hbar\frac{dV}{dx}$$

so

$$\frac{d\langle p\rangle}{dt} = -\left\langle\frac{dV}{dx}\right\rangle$$ again as expected

- What does the commutator mean in a general case? Why is this classical?

+ In Hamiltonian classical mechanics, we define a Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial p}$$ etc

+ You see $\{x,p\} = 1$. Canonical quantization says you get a quantum theory from a classical one by

quantum $\longrightarrow [A,B] = i\hbar\{A,B\} \longleftarrow$ classical

+ In Hamiltonian classical mechanics, observables satisfy $\frac{dg}{dt} = \{g, H\} + \frac{\partial g}{\partial t}$

- Different ways of viewing time dependence
  - In the <u>Schrödinger picture</u>, operators are constant (except for explicit time dependence) and states evolve

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle$$

  - But look at any expectation value

$$\langle \mathcal{O}\rangle(t) = \langle\Psi(0)|e^{iHt/\hbar}\,\mathcal{O}\,e^{-iHt/\hbar}|\Psi(0)\rangle$$

  This suggests:
    + We treat states as time-independent $|\Psi\rangle = |\Psi(0)\rangle$
    + We give time-dependence to operators

$$\mathcal{O}(t) = e^{iHt/\hbar}\,\mathcal{O}(0)\,e^{-iHt/\hbar}$$

  (leaving out explicit time dependence)
    + This is the <u>Heisenberg picture</u> of time-dependence in QM.
  - You will find the "Heisenberg equation" on the homework. Can you guess it from Ehrenfest's theorem?

- <u>Energy-time uncertainty Principle</u>
  - This is a more "squishy" principle than the Heisenberg uncertainty principle
  - Start as usual defining $\Delta E = \sigma_H$.
    + For any operator $\mathcal{O}$, $\sigma_H\sigma_{\mathcal{O}} \geq \frac{1}{2}|\langle[H,\mathcal{O}]\rangle|$
    + The rhs is $\frac{\hbar}{2}\left|\frac{d\langle\mathcal{O}\rangle}{dt}\right|$ (for no explicit time-dependence)
    + Define Δt as the time an operator takes to change "a lot"

$$\frac{d\langle\mathcal{O}\rangle}{dt}\Delta t = \sigma_{\mathcal{O}}$$

  - We get $\Delta E\,\Delta t \geq \hbar/2$. But we can use this in many approximate ways.