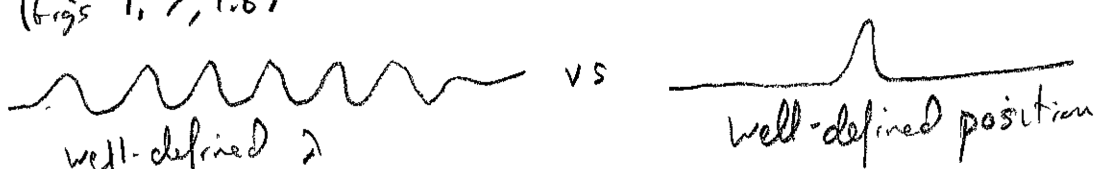
- Heisenberg Uncertainty Principle

This is a very famous and fundamental relationship often written as $\Delta x \Delta p \geq \hbar/2$

— The physical meaning has to do with $\vec{p} = -i\hbar\vec{\nabla}$ as an operator.
That implies, for a $\vec{p}$ eigenstate, $p = 2\pi\hbar/\lambda$, so an error Δp (or spread in measurement) is due to an error $\Delta\lambda$.
But a perfectly localized wave has no set wavelength (Fourier)
so there is a tradeoff between Δx and Δp

- As in text (figs 1.7, 1.8)



- More or less a consequence of deBroglie's ideas on wave-particle duality at a heuristic level

— More precisely, we should take $\Delta x = \sigma_x$, $\Delta p = \sigma_p$, the standard deviation of repeated measurements on an ensemble of identically prepared systems.

- Recall for any operator $\mathcal{O}$ in state $|\psi\rangle$

$$\sigma^2 = \langle \Delta\mathcal{O}^2 \rangle = \langle \psi | (\mathcal{O} - \langle \mathcal{O} \rangle)^2 | \psi \rangle = \langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2$$

For Hermitian $\mathcal{O}$, we can also write this as

$$\sigma^2 = \langle f | f \rangle, \text{ where } |f\rangle = \Delta\mathcal{O}|\psi\rangle.$$

- Look at two observables A and B where $|f\rangle = \Delta A|\psi\rangle$, $|g\rangle = \Delta B|\psi\rangle$

Then

$$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2 \text{ by Schwarz inequality}$$

(Essentially, $\langle f|g\rangle = \sqrt{\langle f|f\rangle}\sqrt{\langle g|g\rangle}\cos\theta$)

+ But $|\langle f|g\rangle|^2 \geq (\mathrm{Im}\langle f|g\rangle)^2 = \frac{-1}{4}(\langle f|g\rangle - \langle g|f\rangle)^2$

+ Put it together

$$\sigma_A^2 \sigma_B^2 \geq \frac{-1}{4}\left[\langle\psi|\Delta A \Delta B|\psi\rangle - \langle\psi|\Delta B \Delta A|\psi\rangle\right]^2$$

What does that mean? Does it vanish?

- Operators do not commute! $AB|\psi\rangle \neq BA|\psi\rangle$ generally

+ As an example $xp|\psi\rangle$ in position basis (ie, sandwich $\langle x|$) is

$$x\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x) = -i\hbar\, x\frac{\partial\psi}{\partial x} \quad \text{vs} \quad [illegible]$$

$$px|\psi\rangle \longrightarrow \left(-i\hbar\frac{\partial}{\partial x}\right)(x\psi) = -i\hbar\, x\frac{\partial\psi}{\partial x} + (-i\hbar)\psi$$

+ We see

$$(xp - px)|\psi\rangle = -i\hbar|\psi\rangle$$ independent of the state

That lets us define the operator commutator $[x,p] = -i\hbar$

+ Generally $[A,B] \equiv AB - BA = C$, another operator

- So

$$\sigma_A^2\sigma_B^2 \geq \frac{-1}{4}\langle\psi|[A,B]|\psi\rangle^2 \quad (\text{Show } [\Delta A, \Delta B] = [A,B])$$

Comments: + $[A,B]^\dagger = -[A,B]$ if A, B Hermitian

Means eigenvalues + expectation values are imaginary

+ The Heisenberg uncertainty principle is $\sigma_x\sigma_p \geq \hbar/2$

+ Our result applies to any operators and is a general uncertainty principle

– Wavefunctions of minimal uncertainty $\sigma_x\sigma_p = \hbar/2$ are Gaussian

$$\psi(x) \propto e^{-ax^2}$$

More on this later.

– We will of course have many more examples in specific systems. For now, the physical interpretation is that measurement of one observable (say position with light) changes the value of another (momentum).